Vafa-Witten Theory: Invariants, Floer Homologies, Higgs Bundles, a Geometric Langlands Correspondence, and Categorification

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Outline of Talk

- Introduction and Motivation
- Summary of Results
- Main Body of the Talk
- Conclusion

Introduction and Motivation

In this talk, we will discuss the **Vafa-Witten (VW) twist** of a 4d $\mathcal{N} = 4$ SYM gauge theory on M_4.

The motivations for doing so are to:

- Derive a **novel VW invariant** of M_4, and relate it to Gromov-Witten (GW) invariants via an $\mathcal{N} = (4, 4)$ A-model.
- Derive a **novel Vafa-Witten Atiyah-Floer correspondence**, and thereby a **physical proof and generalization** of the **Abouzaid-Manolescu conjecture** of hypercohomology of perverse sheaves in [1].
- Obtain a **Langlands dual** of the invariants, Floer homologies and hypercohomology stated hitherto, as well as the quantum and classical Geometric Langlands correspondence [2].
- Obtain a physical framework for **higher categorification** of VW theory.

Introduction and Motivation

This talk is based on

- Ong, Zhi-Cong and Tan, Meng-Chwan, *Vafa-Witten Theory: Invariants, Floer Homologies, Higgs Bundles, a Geometric Langlands Correspondence, and Categorification*, arXiv preprint arXiv:2203.17115 (ATMP in press)

Built on earlier insights in

- Bershadsky, Michael, et al, *Topological reduction of 4D SYM to 2D* σ*-models*, Nuclear Physics B 448.1-2, 166-186 (1995).
- Birmingham, Danny, et al, *Topological field theory*, Physics Reports 209.4-5, 129-340 (1991).
- Gukov, Sergei, *Surface operators and knot homologies*, New Trends in Mathematical Physics, 2009 - Springer.
- Kapustin, Anton and Witten, Edward. *Electric-magnetic duality and the geometric Langlands program*, Communications in Number Theory and Physics Volume 1, Number 1, (2007).

Summary of Results

1. If the scalar curvature of M_4 and the gauge group G are not simultaneously non-negative and locally a product of $SU(2)$'s, the theory localizes on a moduli space of configurations satisfying the VW equations. The invariant is the partition function

$$\boxed{\mathcal{Z}_{\mathsf{VW},M_4}(\tau, G) = \sum_k a_k q^{m_k}, \quad q = e^{2\pi i \tau}}$$

k denotes the k^{th} sector of the moduli space $\mathcal{M}_{\mathsf{VW}}$ of the VW equations. The number a_k is

$$\boxed{a_k = \int_{\mathcal{M}^k_{\mathsf{VW}}} \Omega^0 \wedge e(T_{\mathcal{M}^k_{\mathsf{VW}}}), \quad \text{where} \quad \Omega^0(\mathcal{M}^k_{\mathsf{VW}}) = (1 + B^4)^{dim_{\mathbb{C}} \mathcal{M}^k_{\mathsf{VW}}}}$$

B is a coordinate on $\mathcal{M}^k_{\mathsf{VW}}(A, B)$, e is the signed Euler class of the tangent bundle $T_{\mathcal{M}^k_{\mathsf{VW}}}$.

2. Compactify VW theory on $M_4 = \Sigma \times C$ along C, where both Σ and C are closed Riemann surfaces of genus $g = 1$ and $g \geq 2$, respectively. We arrive at an A-model in complex structure I on Σ with $\mathcal{N} = (4,4)$ supersymmetry and target space $\mathcal{M}_H^G(C)$, the moduli space of Hitchin's equations on C. Topological invariance implies a 4d-2d correspondence

$$\boxed{\mathcal{Z}_{\text{VW},M_4}(\tau, G) = \mathcal{Z}_{\text{GW},\Sigma}(\tau, \mathcal{M}^G_{\text{Higgs}}(C)) = \sum_l \tilde{a}_l q^{\tilde{m}_l}}$$

where l denotes the l^{th} sector of the moduli space $\mathcal{M}_{\text{maps}}$ of holomorphic maps for *genus one*, the rational number $\tilde{a}_l$ is

$$\boxed{\tilde{a}_l = \int_{\mathcal{M}^l_{\text{maps}}} e(\mathcal{V})}$$

where e is the signed Euler class of the vector bundle $\mathcal{V}$ with fiber $H^0(\Sigma, K \otimes \Phi^* T^* \mathcal{M}^l_{\text{maps}})$ and canonical bundle K on Σ.

Summary of Results

3. Boundary VW theory on $M_4 = M_3 \times \mathbb{R}^+$, with M_3 a closed three-manifold in the temporal gauge allows us to recast the 4d theory as 1d supersymmetric quantum mechanics (SQM) on the space of all **complexified** connections on M_3, with potential being the **complex** Chern-Simons functional. The VW partition function is then

$$\boxed{\mathcal{Z}_{\text{VW},M_4}(\tau, G) = \sum_k \mathcal{F}^{G,\tau}_{\text{VW}}(\Psi^k_{M_3}) = \sum_k \text{HF}^{\text{VW}}_k(M_3, G, \tau) = \mathcal{Z}^{\text{Floer}}_{\text{VW},M_3}(\tau, G)}$$

4. We then Heegaard split M_3 along the Riemann surface C to obtain an equivalent open A-model with boundaries given by Lagrangian (A, B, A)-branes L_0, L_1 in $\mathcal{M}^G_{\text{Higgs}}(C)$, which leads us to a VW Atiyah-Floer correspondence as

$$\boxed{\text{HF}^{\text{VW}}_*(M_3, G, \tau) \cong \text{HF}^{\text{Lagr}}_*\big(\mathcal{M}^G_{\text{Higgs}}(C), L_0, L_1, \tau\big)}$$

Summary of Results

5. This allows us to physically realize the Abouzaid-Manolescu conjecture for the hypercohomology $\mathrm{HP}^*(M_3)$ of a perverse sheaf of vanishing cycles in the moduli space of irreducible flat $SL(2,\mathbb{C})$-connections on M_3, which can be generalized to

$$\boxed{\mathrm{HP}^*(M_3) \cong \mathrm{HF}_*^{\mathrm{inst}}(M_3, G_{\mathbb{C}}, \tau)}$$

6. S-duality of $\mathcal{N}=4$ VW theory implies a Langlands duality of the aforementioned invariants and Floer homologies.

7. Also, when we replace M_3 with $I \times C$ where $C \to 0$, from S-duality, we have a homological mirror symmetry of the category of A-branes

$$\boxed{\mathrm{Cat}_{A\text{-branes}}\left(\tau, \mathcal{M}^{G}_{\mathrm{Higgs}}(C)\right) \longleftrightarrow \mathrm{Cat}_{A\text{-branes}}\left(-\frac{1}{n_{\mathfrak{g}}\tau}, \mathcal{M}^{^LG}_{\mathrm{Higgs}}(C)\right)}$$

8. If $\mathrm{Re}(\tau) = 0$, we obtain a quantum geometric Langlands correspondence

$$\boxed{\mathcal{D}^{\mathbf{c}}_{-h^\vee}\text{-mod}\big(q, \mathsf{Bun}_{G_{\mathbb{C}}}\big) \longleftrightarrow \mathcal{D}^{\mathbf{c}}_{-{}^Lh^\vee}\text{-mod}\Big(-\frac{1}{n_{\mathfrak{g}} q}, \mathsf{Bun}_{{}^LG_{\mathbb{C}}}\Big)}$$

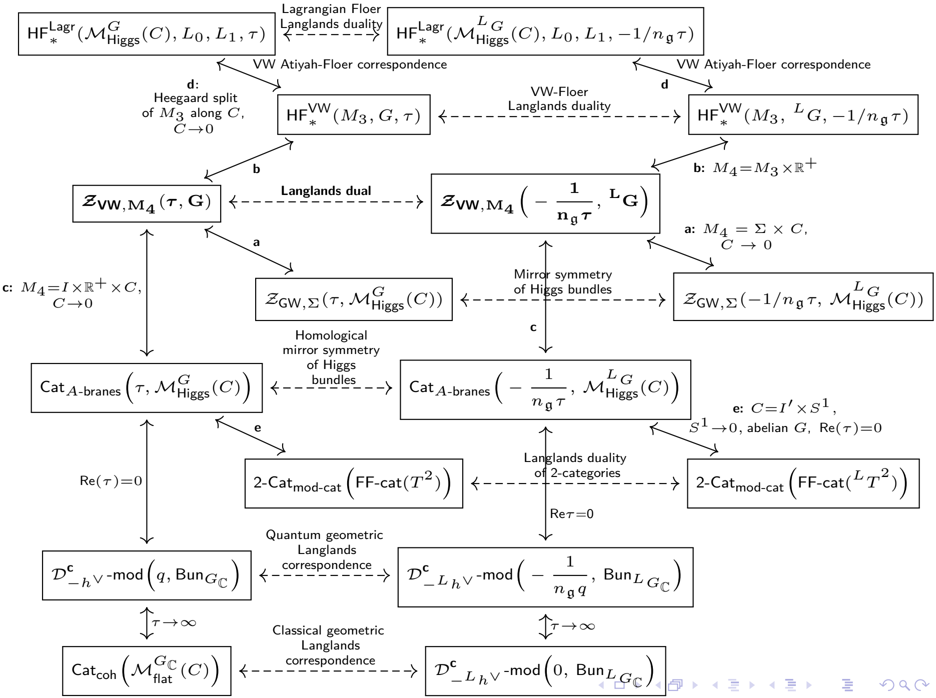
9. In the "classical" $\tau \to \infty$ limit, this becomes the classical geometric Langlands correspondence

$$\boxed{\mathsf{Cat}_{\mathsf{coh}}\big(\mathcal{M}^{G_{\mathbb{C}}}_{\mathsf{flat}}(C)\big) \longleftrightarrow \mathcal{D}^{\mathbf{c}}_{-{}^Lh^\vee}\text{-mod}\Big(0, \mathsf{Bun}_{{}^LG_{\mathbb{C}}}\Big)}$$

10. One can also observe that by successively adding boundaries to the underlying manifold as we have done, the VW invariant will be categorified as

$$\boxed{\mathcal{Z}_{\mathsf{VW}} \xrightarrow{\text{categorification}} \mathsf{HF}^{\mathsf{VW}}_{*} \xrightarrow{\text{categorification}} \mathsf{Cat}_{A\text{-branes}} \xrightarrow{\text{categorification}} \text{2-}\mathsf{Cat}_{\text{mod-cat}}\Big(\text{FF-cat}(T^2)\Big)}$$

The last 2-category is that of module categories over the Fukaya- Floer category of T^2 when we further let $C = I' \times S^1$ where $S^1 \to 0$, and it is assigned to S^1, just like how $\mathsf{Cat}_{A\text{-branes}}$ is assigned to C, $\mathsf{HF}^{\mathsf{VW}}_{*}$ is assigned to M_3, and $\mathcal{Z}_{\mathsf{VW}}$ is assigned to M_4.


$\mathrm{HF}_*^{\mathrm{Lagr}}(\mathcal{M}_{\mathrm{Higgs}}^G(C), L_0, L_1, \tau)$
Lagrangian Floer Langlands duality
$\mathrm{HF}_*^{\mathrm{Lagr}}(\mathcal{M}_{\mathrm{Higgs}}^{^LG}(C), L_0, L_1, -1/n_{\mathfrak{g}}\tau)$
VW Atiyah-Floer correspondence
VW Atiyah-Floer correspondence
d: Heegaard split of M_3 along C, $C \to 0$
$\mathrm{HF}_*^{\mathrm{VW}}(M_3, G, \tau)$
VW-Floer Langlands duality
d
$\mathrm{HF}_*^{\mathrm{VW}}(M_3, {}^LG, -1/n_{\mathfrak{g}}\tau)$
b
b: $M_4 = M_3 \times \mathbb{R}^+$
$\mathcal{Z}_{\mathrm{VW},M_4}(\tau, G)$
Langlands dual
$\mathcal{Z}_{\mathrm{VW},M_4}\left(-\frac{1}{n_{\mathfrak{g}}\tau}, {}^LG\right)$
a
a: $M_4 = \Sigma \times C$, $C \to 0$
c: $M_4 = I \times \mathbb{R}^+ \times C$, $C \to 0$
$\mathcal{Z}_{\mathrm{GW},\Sigma}(\tau, \mathcal{M}_{\mathrm{Higgs}}^G(C))$
Mirror symmetry of Higgs bundles
$\mathcal{Z}_{\mathrm{GW},\Sigma}(-1/n_{\mathfrak{g}}\tau, \mathcal{M}_{\mathrm{Higgs}}^{^LG}(C))$
c
Homological mirror symmetry of Higgs bundles
$\mathrm{Cat}_{A\text{-branes}}\left(\tau, \mathcal{M}_{\mathrm{Higgs}}^G(C)\right)$
$\mathrm{Cat}_{A\text{-branes}}\left(-\frac{1}{n_{\mathfrak{g}}\tau}, \mathcal{M}_{\mathrm{Higgs}}^{^LG}(C)\right)$
e
e: $C = I' \times S^1$, $S^1 \to 0$, abelian G, $\mathrm{Re}(\tau) = 0$
$\mathrm{Re}(\tau) = 0$
$\text{2-Cat}_{\text{mod-cat}}\left(\text{FF-cat}(T^2)\right)$
Langlands duality of 2-categories
$\text{2-Cat}_{\text{mod-cat}}\left(\text{FF-cat}({}^LT^2)\right)$
$\mathrm{Re}\tau = 0$
Quantum geometric Langlands correspondence
$\mathcal{D}^{\mathbf{c}}_{-h^\vee}\text{-mod}\left(q, \mathrm{Bun}_{G_{\mathbb{C}}}\right)$
$\mathcal{D}^{\mathbf{c}}_{-^Lh^\vee}\text{-mod}\left(-\frac{1}{n_{\mathfrak{g}}q}, \mathrm{Bun}_{^LG_{\mathbb{C}}}\right)$
$\tau \to \infty$
$\tau \to \infty$
Classical geometric Langlands correspondence
$\mathrm{Cat}_{\mathrm{coh}}\left(\mathcal{M}_{\mathrm{flat}}^{G_{\mathbb{C}}}(C)\right)$
$\mathcal{D}^{\mathbf{c}}_{-^Lh^\vee}\text{-mod}\left(0, \mathrm{Bun}_{^LG_{\mathbb{C}}}\right)$

LET'S EXPLAIN HOW WE GOT THESE RESULTS

Vafa-Witten Theory on M_4

- VW theory has a **single scalar supercharge** $\mathcal{Q}$, whose BPS equations are obtained by setting the $\mathcal{Q}$-variation of fermions to zero:

$$\begin{aligned} F^+_{\mu\nu} + \frac{1}{2}[B_{\mu\nu}, C] + \frac{1}{4}[B_{\mu\rho}, B_{\lambda\nu}]g^{\rho\lambda} &= 0, \\ \mathcal{D}_\mu C + \mathcal{D}_\nu B^{\nu\mu} &= 0. \end{aligned} \tag{1}$$

- We then set $C = 0$ to ensure that there are no reducible gauge connections A.

- The 2-form B need not vanish if the scalar curvature of M4 and the gauge group G are not simultaneously non-negative and locally a product of $SU(2)$'s, and we will assume this to be the case here.

A Vafa-Witten Invariant of M_4

- VW theory is a balanced TQFT (same number of fermion pair zero modes), and the path integral localizes to the moduli space $\mathcal{M}_{\text{VW}}$, whence the only non-vanishing topological invariant is the partition function

$$\boxed{\mathcal{Z}_{\text{VW},M_4}(\tau, G) = \sum_k a_k q^{m_k}} \tag{2}$$

Here, $q = e^{2\pi i\tau}$, k denotes the k^{th} sector of $\mathcal{M}_{\text{VW}}$, the number a_k is

$$\boxed{a_k = \int_{\mathcal{M}^k_{\text{VW}}} \Omega^0 \wedge e(T_{\mathcal{M}^k_{\text{VW}}})\,, \quad \text{where} \quad \Omega^0(\mathcal{M}^k_{\text{VW}}) = (1 + B^4)^{dim_{\mathbb{C}}\mathcal{M}^k_{\text{VW}}}}$$

B is a coordinate on $\mathcal{M}^k_{\text{VW}}(A, B)$, e is the signed Euler class of the tangent bundle $T_{\mathcal{M}^k_{\text{VW}}}$, and m_k is the corresponding VW number

$$\boxed{m_k = \frac{1}{8\pi^2}\int_{M_4} \text{Tr}\left(F_{(k)} \wedge F_{(k)} + dB_{(k)} \wedge \star DB_{(k)} + B_{(k)} \wedge d(\star DB_{(k)})\right)}$$

$\mathcal{N} = (4, 4)$ A-model, Higgs Bundles and GW Theory

- We consider a block diagonal metric g for $M_4 = \Sigma \times C$, where the genus are $g = 1$ and $g \geq 2$ for Σ and C, respectively.

$$g = \text{diag}(g_\Sigma, \epsilon g_C), \tag{3}$$

  where ϵ is a small parameter to deform g_C.

- When $\epsilon \to 0$, in order for the action to remain well-defined, i.e. finite, we obtain the following conditions along C:

$$F_C - \varphi \wedge \varphi = D\varphi = D^*\varphi = 0. \tag{4}$$

  Here, A_C and a section $\varphi \in \Omega^1(C)$ modulo gauge transformations span **Hitchin's moduli space** $\mathcal{M}_H^G(C)$.

- We get a sigma model on Σ with a map $\Phi(X, Y) : \Sigma \to \mathcal{M}_H^G(C)$, where the bosonic scalars (X, Y) on Σ correspond to (A_C, φ_C).

- The BPS equations of the sigma model are **holomorphic maps**, obtained from the dimensional reduction of (1):

$$\partial_{\bar{z}} X^i = \partial_{\bar{z}} Y^i = 0. \tag{5}$$

$\mathcal{N} = (4,4)$ A-model, Higgs Bundles and GW Theory

- This is an A-model, and it can be further shown that the complex structure is I, whence the target space $\mathcal{M}_H^G(C) = \mathcal{M}_{\text{Higgs}}^G(C)$.

- The topological invariant is the partition function, a GW invariant:

$$\boxed{\mathcal{Z}_{\text{GW},\Sigma}(\tau, \mathcal{M}_{\text{Higgs}}^G(C)) = \sum_l \tilde{a}_l q^{\tilde{m}_l}} \quad (6)$$

l denotes the l^{th} sector of $\mathcal{M}_{\text{maps}}$ for *genus one* Σ, the rational number $\tilde{a}_l$ is given by [6]

$$\boxed{\tilde{a}_l = \int_{\mathcal{M}_{\text{maps}}^l} e(\mathcal{V})}$$

where e is the signed Euler class of the vector bundle $\mathcal{V}$ with fiber $H^0(\Sigma, K \otimes \Phi^* T^* \mathcal{M}^l_{\text{maps}})$ and canonical bundle K on Σ, and $\tilde{m}_l$ is the corresponding worldsheet instanton number given by

$$\boxed{\tilde{m}_l = \frac{1}{2\pi} \int_\Sigma \Phi_l^*(\omega_I)}$$

- Topological invariance implies a **4d-2d correspondence**:

$$\boxed{\mathcal{Z}_{\text{VW},M_4}(\tau, G) = \mathcal{Z}_{\text{GW},\Sigma}(\tau, \mathcal{M}^G_{\text{Higgs}}(C))} \tag{7}$$

  between the VW invariant of $M_4 = \Sigma \times C$ and the GW invariant of $\mathcal{M}^G_{\text{Higgs}}(C))$.

- Let $M_4 = M_3 \times \mathbb{R}^+$, where the M_3 boundary is a closed three-manifold, and $\mathbb{R}^+$ is the 'time' coordinate. Using the temporal gauge $A^0 = 0$, and exploiting the self-duality of the 2-form $B_{\mu\nu}$, VW equations in (1) become

$$\begin{aligned} \dot{A}^i + \frac{1}{2}\epsilon^{ijk}(F_{jk} - [B_j, B_k]) &= 0, \\ \dot{B}^i + \epsilon^{ijk}(\partial_j B_k + [A_j, B_k]) &= 0. \end{aligned} \tag{8}$$

- If we define a complexified connection $\mathcal{A} = A + iB \in \Omega^1(M_3)$, of a $G_{\mathbb{C}}$-bundle on M_3, we can rewrite (8) as a **flow equation**:

$$\frac{d\mathcal{A}^i}{dt} + s g^{ij}_{\mathfrak{A}} \frac{\partial V(\mathcal{A})}{\partial \mathcal{A}^j} = 0. \tag{9}$$

- $\mathfrak{A}$ is the space of complexified connections $\mathcal{A}$ with metric $g^{ij}_{\mathfrak{A}}$, s is a tuneable parameter, and

$$V(\mathcal{A}) = -\frac{1}{4\pi^2}\int_{M_3} \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big). \tag{10}$$

1d SQM from Boundary Vafa-Witten Theory

- The 4d action of boundary VW theory can be rewritten as a **1d SQM model** with target space $\mathfrak{A}$:

$$S_{\text{VW}}^{\text{bdry}} = \frac{1}{e^2} \int dt \left| \frac{d\mathcal{A}^i}{dt} + s g_{\mathfrak{A}}^{ij} \frac{\partial V(\mathcal{A})}{\partial \mathcal{A}^j} \right|^2 + \dots \qquad (11)$$

- The partition function of boundary VW theory will localize onto the configurations that minimize the above term.
- That is, it will be an **algebraic count of time-invariant critical points of the complex Chern-Simons functional**, corresponding to flat $G_{\mathbb{C}}$-connections on M_3.
- The complex Chern-Simons functional $V(\mathcal{A})$ is a Morse functional, whose **critical points generate a Floer complex**.
- VW flow lines between critical points, as described by the gradient flow equation (9), can be interpreted as Floer differentials, whence **the number of outgoing flow lines at each critical point is the degree of the corresponding chain in the complex**.

- The partition function of boundary VW theory is originally expressed as

$$\mathcal{Z}_{\mathsf{VW},M_4}(\tau,G)=\int_{\mathcal{M}_{\mathsf{VW}}}\mathcal{F}(\Psi_{M_3})\,e^{-S^{\mathsf{bdry}}_{\mathsf{VW}}}=\sum_k\left\langle\mathcal{F}^{G,\tau}_{\mathsf{VW}}(\Psi^k_{M_3})\right\rangle. \quad (12)$$

- By comparing with the results from the 1d SQM perspective, we have

$$\mathcal{F}^{G,\tau}_{\mathsf{VW}}(\Psi^k_{M_3})\in\mathsf{HF}^{\mathsf{VW}}_k(M_3,G,\tau), \quad (13)$$

where the HF's are the **VW Floer homology classes**.

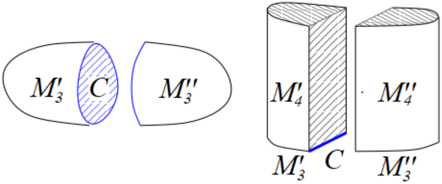
- Thus, boundary VW theory allows us to define a novel VW Floer homology assigned to M_3 as

$$\boxed{\mathcal{Z}_{\mathsf{VW},M_4}(\tau,G)=\sum_k\mathcal{F}^{G,\tau}_{\mathsf{VW}}(\Psi^k_{M_3})=\sum_k\mathsf{HF}^{\mathsf{VW}}_k(M_3,G,\tau)=\mathcal{Z}^{\mathsf{Floer}}_{\mathsf{VW},M_3}(\tau,G)} \quad (14)$$

where the second and third expressions are understood to be expectation values of operators.

A Vafa-Witten Atiyah-Floer Correspondence

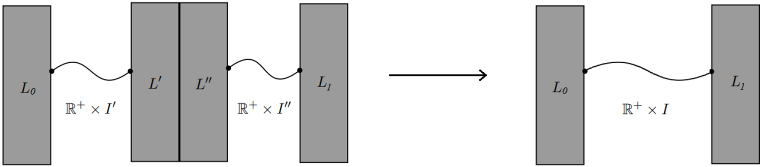
- We perform a Heegaard split of $M_3 = M_3' \cup_C M_3''$ along C by writing $M_4 = (\mathbb{R}^+ \times I' \times C) \cup_C (\mathbb{R}^+ \times I'' \times C)$:



- When $C \to 0$, we have an open A-model in complex structure I on $\mathbb{R}^+ \times I'$ and $\mathbb{R}^+ \times I''$, with target space $\mathcal{M}^G_{\text{Higgs}}(C)$.
- These describe open strings with worldsheets $\mathbb{R}^+ \times I'$ and $\mathbb{R}^+ \times I''$ that propagate in $\mathcal{M}^G_{\text{Higgs}}(C)$ and end on A-branes. Specifically, we have a certain (A, B, A)-brane that is an A-brane in $\mathcal{M}^G_H(C)$ in complex structure K, that will correspond to **flat $G_{\mathbb{C}}$-connections on C which can be extended to $M_3^{','' }$**, as required.

A Vafa-Witten Atiyah-Floer Correspondence

- With two split pieces M_4' and M_4'', when $C \to 0$, we have two strings, each ending on pairs of Lagrangian branes (L_0, L') and (L'', L_1).



- Gluing the open worldsheets together along their common boundary L' and L'' gives a single A-model on $\mathbb{R}^+ \times I$, with a single string extending from L_0 to L_1, which is equivalent to gluing M_4' and M_4'' along $C \times \mathbb{R}^+$.
- Next, we recast the A-model here as an SQM model, where the target space is $\mathscr{P}(L_0, L_1)$, the **space of smooth trajectories from L_0 to L_1** (arising from the interval I that connects them).

A Vafa-Witten Atiyah-Floer Correspondence

- The 2d BPS equation for this open A-model can be written as a **gradient flow equation** for the 1d SQM model

$$\frac{\partial Z^l}{\partial t} + i\frac{\partial Z^l}{\partial s} = 0. \tag{15}$$

- Critical points correspond to **stationary trajectories in** $\mathscr{P}(L_0, L_1)$, i.e., the intersection points of L_0 and L_1, which generate the chains of the Lagrangian Floer complex.
- Intersection points belong to Lagrangian Floer homology classes

$$\sum_i (L_0 \cap L_1)^n_i \in \mathrm{HF}^{\mathrm{Lagr}}_n(\mathcal{M}^G_{\mathrm{Higgs}}(C), L_0, L_1). \tag{16}$$

- Floer differentials are the **outgoing flow lines at each** $L_0 \cap L_1$**, which number would be the degree of the corresponding chain in the complex**.

A Vafa-Witten Atiyah-Floer Correspondence

- The partition function of the open A-model is then given by

$$\mathcal{Z}_{A,L}(\tau, \mathcal{M}^{G}_{\text{Higgs}}(C)) = \sum_{n} \text{HF}^{\text{Lagr}}_{n}(\mathcal{M}^{G}_{\text{Higgs}}(C), L_0, L_1, \tau). \qquad (17)$$

- Topological invariance gives an **equivalence of partition functions**, which implies

$$\sum_{k} \text{HF}^{\text{VW}}_{k}(M_3, G, \tau) = \sum_{n} \text{HF}^{\text{Lagr}}_{n}(\mathcal{M}^{G}_{\text{Higgs}}(C), L_0, L_1, \tau). \qquad (18)$$

- It can be shown that we can identify the k and n indices, whence we have

$$\boxed{\text{HF}^{\text{VW}}_{*}(M_3, G, \tau) \cong \text{HF}^{\text{Lagr}}_{*}(\mathcal{M}^{G}_{\text{Higgs}}(C), L_0, L_1, \tau)} \qquad (19)$$

This gives a Vafa-Witten Atiyah-Floer Correspondence.

- (A, B, A)-branes of the open A-model can be interpreted as Lagrangian branes in $\mathcal{M}_H^G(C)$ in complex structure K i.e., $\mathcal{M}_{\text{flat}}^{G_{\mathbb{C}}}(C)$, the moduli space of irreducible flat $G_{\mathbb{C}}$-connections on C. The **VW Atiyah-Floer correspondence is then the Atiyah-Floer correspondence for $G_{\mathbb{C}}$-instantons.**

$$\boxed{\text{HF}_*^{\text{inst}}(M_3, G_{\mathbb{C}}, \tau) \cong \text{HF}_*^{\text{Lagr}}(\mathcal{M}_{\text{flat}}^{G_{\mathbb{C}}}(C), L_0, L_1, \tau)} \tag{20}$$

- $\text{HP}^*(M_3)$ is the hypercohomology of the perverse sheaf of vanishing cycles in $X_{\text{irr}}(C)$, the moduli space of irreducible flat $SL(2, \mathbb{C})$-connections on C.

- There is a one-to-one correspondence between the gradings of $\text{HP}^*(M_3)$ and $\text{HF}_*^{\text{Lagr}}$, and the chains underlying their complexes, whence we can identify

$$\text{HP}^*(M_3) \cong \text{HF}_*^{\text{Lagr}}(X_{\text{irr}}(C), L_0, L_1, \tau), \tag{21}$$

in agreement with [3, Remark 6.15].

The Abouzaid-Manolescu Conjecture and its Generalization

- According to (20) and (21), for $G_{\mathbb{C}} = SL(2, \mathbb{C})$, we have

$$\boxed{\mathsf{HP}^*(M_3) \cong \mathsf{HF}^{\mathsf{inst}}_*(M_3, SL(2, \mathbb{C}), \tau)} \tag{22}$$

  This is just the the Abouzaid-Manolescu conjecture in [1].

- The Atiyah-Floer correspondence for $G_{\mathbb{C}}$ instantons in (20), and a $G_{\mathbb{C}}$ generalization of (21) in [3, Remark 6.15], imply that a generelization of the Abouzaid-Manolescu conjecture is also possible as

$$\boxed{\mathsf{HP}^*(M_3, G_{\mathbb{C}}) \cong \mathsf{HF}^{\mathsf{inst}}_*(M_3, G_{\mathbb{C}}, \tau)} \tag{23}$$

- $\mathcal{N} = 4$ supersymmetric Yang-Mills theories have an $SL(2,\mathbb{Z})$ (or a subgroup thereof) symmetry on τ relating a theory to its S-dual, giving

$$\mathcal{Z}_{\mathsf{VW},M_4}(\tau, G) \longleftrightarrow \mathcal{Z}_{\mathsf{VW},M_4}\Big(-\frac{1}{n_{\mathfrak{g}}\tau}, {}^L G\Big)$$

$$\mathcal{Z}_{GW,\Sigma}(\tau, \mathcal{M}^{G}_{\mathsf{Higgs}}(C)) \longleftrightarrow \mathcal{Z}_{GW,\Sigma}\Big(-\frac{1}{n_{\mathfrak{g}}\tau}, \mathcal{M}^{^L G}_{\mathsf{Higgs}}(C)\Big)$$

$$\mathsf{HF}^{\mathsf{VW}}_{*}(M_3, G, \tau) \longleftrightarrow \mathsf{HF}^{\mathsf{VW}}_{*}(M_3, {}^L G, -1/n_{\mathfrak{g}}\tau)$$

$$\mathsf{HF}^{\mathsf{Lagr}}_{*}(\mathcal{M}^{G}_{\mathsf{Higgs}}(C), L_0, L_1, \tau) \longleftrightarrow \mathsf{HF}^{\mathsf{Lagr}}_{*}(\mathcal{M}^{^L G}_{\mathsf{Higgs}}(C), L_0, L_1, -1/n_{\mathfrak{g}}\tau)$$

$$\mathsf{HP}^{*}(M_3, G_{\mathbb{C}}, \tau) \longleftrightarrow \mathsf{HP}^{*}(M_3, {}^L G_{\mathbb{C}}, -1/n_{\mathfrak{g}}\tau)$$

A Quantum and Classical Geometric Langlands Correspondence

- If we let $M_4 = I \times \mathbb{R}^+ \times C$ with $C \to 0$, S-duality gives a homological mirror symmetry of the category of A-branes

$$\boxed{\text{Cat}_{A\text{-branes}}(\tau, \mathcal{M}^{G}_{\text{Higgs}}(C)) \longleftrightarrow \text{Cat}_{A\text{-branes}}\Big(-\frac{1}{n_{\mathfrak{g}}\tau}, \mathcal{M}^{^LG}_{\text{Higgs}}(C)\Big)} \tag{24}$$

- When $\text{Re}(\tau) = 0$, the category of A-branes can be identified with a category of twisted D-modules on $\text{Bun}_{G_{\mathbb{C}}}(C)$ with parameter $q = \tau$ [4], whence we have

$$\boxed{\mathcal{D}^{\mathbf{c}}_{-h^\vee}\text{-mod}(q, \text{Bun}_{G_{\mathbb{C}}}) \longleftrightarrow \mathcal{D}^{\mathbf{c}}_{-^Lh^\vee}\text{-mod}\Big(-\frac{1}{n_{\mathfrak{g}}q}, \text{Bun}_{^LG_{\mathbb{C}}}\Big)} \tag{25}$$

a quantum geometric Langlands correspondence [2]. In the "classical limit" of $q \to \infty$, we have the classical correspondence

$$\boxed{\text{Cat}_{\text{coh}}(\mathcal{M}^{G_{\mathbb{C}}}_{\text{flat}}(C)) \longleftrightarrow \mathcal{D}^{\mathbf{c}}_{-^Lh^\vee}\text{-mod}\Big(0, \text{Bun}_{^LG_{\mathbb{C}}}\Big)} \tag{26}$$

Categorification of Vafa-Witten Theory

- Categorification is naturally realized in our physical framework:

$$
\begin{array}{llll}
\text{VW theory on } M_4 & \rightsquigarrow & \text{number} & \mathcal{Z}_{\text{VW}} \\
\text{VW theory on } \mathbb{R}^+ \times M_3 & \rightsquigarrow & \text{vector} & \text{HF}_*^{\text{VW}} \\
\text{VW theory on } \mathbb{R}^+ \times I \times C & \rightsquigarrow & \text{1-category} & \text{Cat}_{A\text{-branes}} \\
\text{VW theory on } \mathbb{R}^+ \times I \times I' \times S^1 & \rightsquigarrow & \text{2-category} & \text{2-Cat} \\
\text{VW theory on } \mathbb{R}^+ \times I \times I' \times [0,1] & \rightsquigarrow & \text{3-category} & \text{3-Cat}
\end{array} \quad . \tag{27}
$$

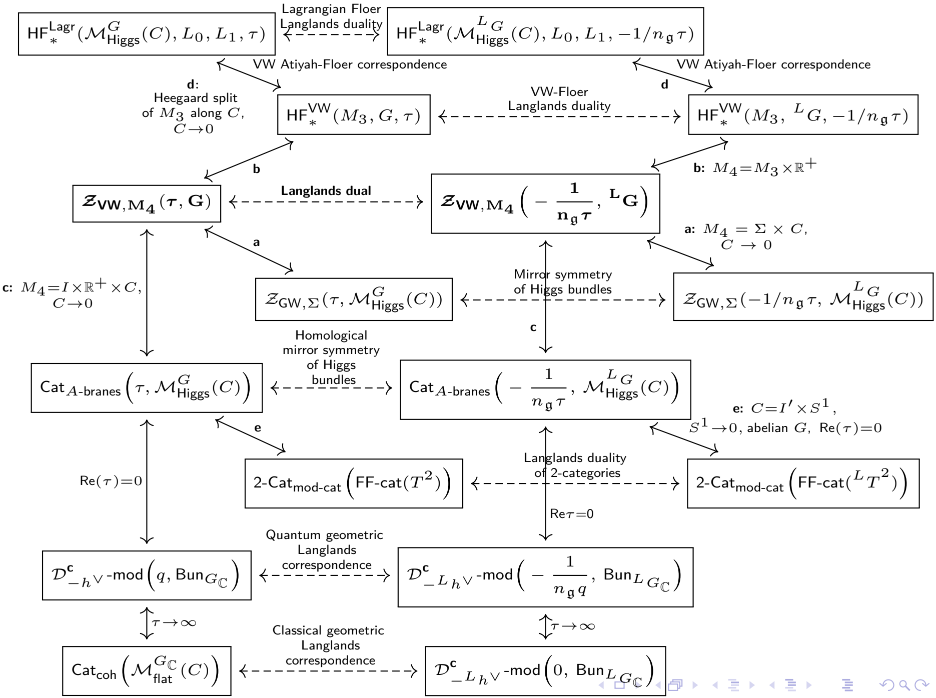
- Notice from our discussion hitherto that as we go down the list, the categories get assigned to M_3, C, ..., and are determined by the category of boundaries of the effective 1d, 2d, ... theory on $\mathbb{R}^+$, $\mathbb{R}^+ \times I$, ...
- Therefore, 2-Cat will be the 2-category of 2d boundaries of the 3d theory on $\mathbb{R}^+ \times I \times I'$ given by VW theory compactified on S^1, that is assigned to S^1.

Categorification of Vafa-Witten Theory

- These 2d boundaries can be realized as surface defects.
- For abelian G and $\text{Re}(\tau) = 0$, the category of such surface defects is the 2-category $\text{2-Cat}_{\text{mod-cat}}(\text{FF-cat}(T^2))$ of module categories over the Fukaya-Floer category of T^2 [5].
- S-duality of VW theory then implies that they enjoy a Langlands duality

$$\boxed{\text{2-Cat}_{\text{mod-cat}}(\text{FF-cat}(T^2)) \longleftrightarrow \text{2-Cat}_{\text{mod-cat}}(\text{FF-cat}({}^L T^2))} \qquad (28)$$

  where ${}^L T^2$ is the dual torus with the radii of the circles inverted.
- Similarly, the 3-Cat will be the 3-category of 3d boundaries of VW theory on $\mathbb{R}^+ \times I \times I' \times [0,1]$ compactified on two points, that is assigned to a point.


$\mathrm{HF}^{\mathrm{Lagr}}_*(\mathcal{M}^G_{\mathrm{Higgs}}(C), L_0, L_1, \tau)$
Lagrangian Floer Langlands duality
$\mathrm{HF}^{\mathrm{Lagr}}_*(\mathcal{M}^{^LG}_{\mathrm{Higgs}}(C), L_0, L_1, -1/n_{\mathfrak{g}}\tau)$
VW Atiyah-Floer correspondence
VW Atiyah-Floer correspondence
d: Heegaard split of M_3 along C, $C \to 0$
d
$\mathrm{HF}^{\mathrm{VW}}_*(M_3, G, \tau)$
VW-Floer Langlands duality
$\mathrm{HF}^{\mathrm{VW}}_*(M_3, {}^LG, -1/n_{\mathfrak{g}}\tau)$
b
b: $M_4 = M_3 \times \mathbb{R}^+$
$\mathcal{Z}_{\mathrm{VW},M_4}(\tau, G)$
Langlands dual
$\mathcal{Z}_{\mathrm{VW},M_4}\left(-\frac{1}{n_{\mathfrak{g}}\tau}, {}^LG\right)$
a
a: $M_4 = \Sigma \times C$, $C \to 0$
c: $M_4 = I \times \mathbb{R}^+ \times C$, $C \to 0$
$\mathcal{Z}_{\mathrm{GW},\Sigma}(\tau, \mathcal{M}^G_{\mathrm{Higgs}}(C))$
Mirror symmetry of Higgs bundles
$\mathcal{Z}_{\mathrm{GW},\Sigma}(-1/n_{\mathfrak{g}}\tau, \mathcal{M}^{^LG}_{\mathrm{Higgs}}(C))$
c
Homological mirror symmetry of Higgs bundles
$\mathrm{Cat}_{A\text{-branes}}\left(\tau, \mathcal{M}^G_{\mathrm{Higgs}}(C)\right)$
$\mathrm{Cat}_{A\text{-branes}}\left(-\frac{1}{n_{\mathfrak{g}}\tau}, \mathcal{M}^{^LG}_{\mathrm{Higgs}}(C)\right)$
e: $C = I' \times S^1$, $S^1 \to 0$, abelian G, $\mathrm{Re}(\tau) = 0$
e
$\mathrm{Re}(\tau) = 0$
$\text{2-Cat}_{\text{mod-cat}}\left(\text{FF-cat}(T^2)\right)$
Langlands duality of 2-categories
$\text{2-Cat}_{\text{mod-cat}}\left(\text{FF-cat}({}^LT^2)\right)$
$\mathrm{Re}\tau = 0$
Quantum geometric Langlands correspondence
$\mathcal{D}^{\mathbf{c}}_{-h^\vee}\text{-mod}\left(q, \mathrm{Bun}_{G_{\mathbb{C}}}\right)$
$\mathcal{D}^{\mathbf{c}}_{-^Lh^\vee}\text{-mod}\left(-\frac{1}{n_{\mathfrak{g}}q}, \mathrm{Bun}_{^LG_{\mathbb{C}}}\right)$
$\tau \to \infty$
$\tau \to \infty$
Classical geometric Langlands correspondence
$\mathrm{Cat}_{\mathrm{coh}}\left(\mathcal{M}^{G_{\mathbb{C}}}_{\mathrm{flat}}(C)\right)$
$\mathcal{D}^{\mathbf{c}}_{-^Lh^\vee}\text{-mod}\left(0, \mathrm{Bun}_{^LG_{\mathbb{C}}}\right)$

Conclusion

- We have physically derived a novel VW invariant of M_4.
- We have a 4d-2d correspondence between VW invariants and GW invariants.
- We have recast boundary VW theory as a 1d SQM model, thereby physically deriving a novel Vafa-Witten Floer homology.
- We went further to physically derive a Vafa-Witten Atiyah-Floer correspondence, which in turn allowed us to physically realize and generalize the Abouzaid-Manolescu conjecture in [1].
- S-duality of VW theory allowed us to obtain Langlands duals of all the invariants and Floer homologies. In certain cases, we could also obtain a quantum and classical geometric Langlands correspondence.
- Our physical framework also allows for a higher categorification of VW theory, whence S-duality again implies a Langlands duality of the relevant higher categories.

THANKS FOR LISTENING!

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