Category of QFTs and differential cohomology

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The goal of this talk

Differential cohomology is a mathematical framework which refines generalized cohomology with differential geometric data on manifolds. They are deeply related with physics.
I would like to understand why they should be related, focusing on the relations with invertible QFTs. We also want to understand the relations in *higher* level.

The plan of this talk :

- As a first (non-extend) answer, I explain the relation with partition functions of invertible theories and differential cohomology. (joint with Yonekura [YY21])
- Next I explain the idea to understand *higher* relations. QFTs should form sheaves of higher categories. We explain an attempt to relate it to the modern framework of differential cohomology in terms of sheaves. (ongoing joint work with K. Ohmori)

Generalized cohomology theory E assigns

$$X: \text{ a topological space} \mapsto E^*(X): \text{an abelian group}$$

$E^*(X)$ has topological information of X.
For example we get *characteristic classes*, or *primary invariants*, such as Chern classes and index of Dirac operators.

Differential cohomology theory $\widehat{E}$ is a refinement of E, assigning

$$X: \text{ a manifold} \mapsto \widehat{E}^*(X): \text{an abelian group}$$

$\widehat{E}^*(X)$ has differential information on X, as well as topological information.
It accounts for secondary invariants, such as holonomy for $U(1)$-connections, Chern-Simons invariants and eta invariants.

Examples of differential cohomologies are,

- $H^2(X;\mathbb{Z}) \simeq \{L \to X : \text{Line bundle}\}/\sim_{\mathrm{isom}}$.
  $\widehat{H}^2(X;\mathbb{Z}) \simeq \{(L,\nabla) \to X : \text{Line bundle with connection}\}/\sim_{\mathrm{isom}}$.
- $K^0(X)$ classifies hermitian vector bundles E over X.
  $\widehat{K}^0(X)$ classifies hermitian vector bundles with connection (E,∇) over X [FL10].

We have commutative diagrams

$$\begin{tikzcd} \widehat{H}^2(X;\mathbb{Z}) \arrow[r, "\mathrm{fgt}"] \arrow[d, "\mathrm{curv}/2\pi i"] & H^2(X;\mathbb{Z}) \arrow[d, "\otimes\mathbb{R}"] \\ \Omega^2_{\mathrm{clo}}(X) \arrow[r, "\mathrm{Rham}"] & H^2(X;\mathbb{R}) \end{tikzcd} \qquad \begin{tikzcd} \widehat{K}^0(X) \arrow[r, "\mathrm{fgt}"] \arrow[d, "\mathrm{ch}"] & K^0(X) \arrow[d, "\mathrm{ch}"] \\ \Omega^{2\mathbb{Z}}_{\mathrm{clo}}(X) \arrow[r, "\mathrm{Rham}"] & H^{2\mathbb{Z}}(X;\mathbb{R}) \end{tikzcd}$$

They are not pullback square, rather a kind of homotopy pullbacks.

There are several models and interpretations for differential cohomologies.

- The idea of homotopy pullback is realized by Hopkins-Singer [HS05]. They constructed a differential extension $\widehat{E}^*$ for each cohomology theory E^*, in terms of *differential function spectra*.
- For $\widehat{H}^n(X;\mathbb{Z})$, we have Cheeger-Simons' model by *differential characters*, in terms of $\mathbb{R}/\mathbb{Z}$-valued functions. The relation with secondary characteristic classes is direct in this model.
- Physically, $\widehat{H}^n(X;\mathbb{Z})$ classifies $(n-1)$-form gauge fields.
- $\widehat{K}$ and $\widehat{KO}$ classifies Ramond-Ramond fields in superstring theories [FMS07] [Fre00].
- In Gomi-Y [GY21] we developed a model of $\widehat{K}$ and $\widehat{KO}$ in terms of fermionic mass terms. Also see Choi-Ohmori [CO22].

Invertible QFTs and generalized cohomology

I only talk about reflection positive and Wick rotated unitary QFTs.
In non-extended case, a QFT $\mathcal{T}$ is called *invertible* if it factors as

$$\mathcal{T}\colon \mathrm{Bord}_d^{\mathcal{S}} \to \mathrm{sLine}_{\mathbb{C}} \subset \mathrm{sVect}_{\mathbb{C}}.$$

In extended case, the condition becomes

$$\mathcal{T}\colon \mathrm{Bord}_d^{\mathcal{S}} \to \mathcal{C}^{\times} \subset \mathcal{C},$$

where $\mathcal{C}^{\times}$ is the maximal Picard ∞-groupoid of the target category $\mathcal{C}$. (Picard groupoid = symmetric monoidal groupoid where all objects are invertible under $\otimes$)

Invertible QFTs are important because they appear as

- SPT (Symmetry Protected Topological) phases in condensed matter physics.
- "Anomaly theories" which describes anomalies of $(d-1)$-dimensional anomalous theories.

Fully extended invertible QFTs are expected to be classified by generalized cohomology theories. The argument is the following [FH21].
The stable homotopy hypothesis states the equivalence

$$(\text{Picard } \infty\text{-groupoids}) \simeq (\text{connective spectra}). \tag{1}$$

A symmetric monoidal functor $\mathcal{T} \colon \mathrm{Bord}_d^{\mathcal{S}} \to \mathcal{C}^\times$ factors through the groupoid completion $|\mathrm{Bord}_d^{\mathcal{S}}|$.
Via the equivalence (1), $\mathcal{T}$ is regarded as a map of spectra, thus giving a generalized cohomology element.

In the case $\mathcal{S} =$ tangential G-structure ($G = \mathrm{SO}, \mathrm{Spin}$ etc), $|\mathrm{Bord}_d^{\mathcal{S}}|$ corresponds to Madsen-Tillmann spectrum MTG.
Freed-Hopkins [FH21] specified the universal target for TQFTs. They conjecture that, in general including non-topological QFTs, the classification is given in terms of Anderson duals.

The Anderson dual

We have $H^n(X;\mathbb{R}) \simeq \mathrm{Hom}(\pi_n(X);\mathbb{R})$.
Also the assignment $X \mapsto \mathrm{Hom}(\pi_*(X);\mathbb{R}/\mathbb{Z})$ is a generalized cohomology, represented by the spectrum $I(\mathbb{R}/\mathbb{Z})$.
Anderson dual to sphere $I\mathbb{Z}$ is the spectrum defined as the homotopy fiber

$$I\mathbb{Z} \to H\mathbb{R} \to I(\mathbb{R}/\mathbb{Z}).$$

This implies that, for any spectrum E, its *Anderson dual* $IE := F(E, I\mathbb{Z})$ fits into the exact sequence (Universal Coefficient Theorem)

$$0 \to \mathrm{Ext}(E_n(X), \mathbb{Z}) \to IE^{n+1}(X) \to \mathrm{Hom}(E_{n+1}(X), \mathbb{Z}) \to 0.$$

MTG represents the G-bordism homology theory Ω^G_*. The Anderson dual $I\Omega^G = F(MTG, I\mathbb{Z})$ fits into the exact sequence

$$0 \to \mathrm{Ext}(\Omega^G_n(X), \mathbb{Z}) \to (I\Omega^G)^{n+1}(X) \to \mathrm{Hom}(\Omega^G_{n+1}(X), \mathbb{Z}) \to 0.$$

We have the exact sequence

$$0 \to \mathrm{Ext}(\Omega^G_n(X), \mathbb{Z}) \to (I\Omega^G)^{n+1}(X) \to \mathrm{Hom}(\Omega^G_{n+1}(X), \mathbb{Z}) \to 0.$$

The interpretation is

- $\mathrm{Ext}(\Omega^G_n(X), \mathbb{Z}) = \mathrm{Hom}(\Omega^G_n(X), \mathbb{R}/\mathbb{Z})/\mathrm{Hom}(\Omega^G_n(X), \mathbb{R})$ classifies invertible n-dimensional TQFTs with G-structure with map to X [FH21].
  Indeed for such a theory $\mathcal{T}$ the partition functions are bordism invariant, so we get $\frac{\arg}{2\pi} Z_{\mathcal{T}} \in \mathrm{Hom}(\Omega^G_n(X), \mathbb{R}/\mathbb{Z})$.
  Dividing by $\mathrm{Hom}(\Omega^G_n(X), \mathbb{R})$ corresponds to take deformation classes.
- To classify possibly non-topological theories we need a larger group. It is conjectured to be $(I\Omega^G)^{n+1}(X)$.

A differential model of $\widehat{IE}$

Motivated by the classification of invertible QFTs, we constructed a "QFT-like" model of differential extension $\widehat{I\Omega^G}$ of $I\Omega^G$ (Yonekura-Y, [YY21]).
The construction works for arbitrary IE, as

$$\widehat{E}_* : \text{ a differential } \textit{homology} \mapsto \widehat{IE}^* : \text{ Anderson dual cohomology}$$

It turns out to be a generalization of Cheeger-Simons' differential characters.

Differential homology

We can easily give a *homology* version of the axiom of differential cohomology, using currents Ω_* instead of differential forms Ω^*.

- For K-homology, we can refine the Baum-Douglas' geometric model. Banameur-Maghfoul [BM06] defines $\widehat{K}_*(X)$ in terms of differential K-cycles (M, E, ∇, f), consisting of Spin^c-manifold M, a hermitian vector bundle with connection $(E, \nabla) \to M$, and a smooth map $f\colon M \to X$. (This is related to D-branes in superstring theories.)
- For bordism homology Ω^G_*, we can define $\widehat{\Omega^G_*}(X)$ in terms of *differential stable G-cycles* (M, g, f), consisting of a manifold M, a G-structure with connection g, and a smooth map $f\colon M \to X$. (The bordism relation use Chern-Weil currents.)
- For ordinary homology $H\mathbb{Z}_*$, we can simply use *differential chains*. $\widehat{H}_n(X)$ is represented by $(c, h, \omega) \in Z^\infty_n(X;\mathbb{Z}) \times C^\infty_{n+1}(X;\mathbb{R}) \times \Omega^{\mathrm{clo}}_n(X)$ with $\partial h = \omega - c$.

$\widehat{IE}^*(X)$

Given a differential homology $\widehat{E}_*$, we can construct a model of differential Anderson dual cohomology $\widehat{IE}^*$. The idea is the following.

Differential homology comes from (cosheaf of) categories $\mathcal{C}_\nabla$ with differential data, for example
$\mathcal{C}_\nabla(X) = \mathrm{Bord}_n^{G_\nabla}(X)$, (differential K-cycles), (differential cycles).
On morphisms of these categories, we can integrate differential forms.

Differential Anderson dual $\widehat{IE}^*(X)$ classifies functors

$$(\omega, h) \colon \mathcal{C}_\nabla(X) \to (\mathbb{R} \to \mathbb{R}/\mathbb{Z})$$

which reflects the differential information.
$h \colon \mathrm{Obj}(\mathcal{C}_\nabla) \to \mathbb{R}/\mathbb{Z}$ is regarded as the phase of a partition function.
$\omega \colon \mathrm{Mor}(\mathcal{C}_\nabla) \to \mathbb{R}$ should be given by integration of differential forms.
In gereral, we use $\mathcal{C}_\nabla(X) = \left(\Omega_n(X; V_\bullet^E)/\mathrm{Im}(\partial) \to \widehat{E}_{n-1}(X)\right)$.

In the case of G-bordism theory, the above definition becomes,

Definition 2 ([YY21])

We define $(\widehat{I\Omega^G})^{d+1}(X) := \{(\omega, h)\}$, where

- $\omega \in \Omega^{d+1}_{\mathrm{clo}}(X; (\mathrm{Sym}\mathfrak{g}^*)^G)$,
- h is a function assigning $h(M^d, g, f) \in \mathbb{R}/\mathbb{Z}$ to each closed d-dimensional $G_\nabla \times X$-manifold. We require the additivity under disjoint unions.

Such that, for any compact $G_\nabla \times X$-manifold (W^{d+1}, g, f) with boundary, we have

$$h(\partial W, g|_{\partial W}, f|_{\partial W}) \equiv \int_W \mathrm{cw}_g(f^*\omega) \pmod{\mathbb{Z}}.$$

The interpretations are,

- h is the complex phase of partition functions of an invertible QFT.
- ω is the characteristic polynomial/forms on target, which measures the variation of h under bordisms.

Category of QFTs and differential cohomology

We have seen that invertible QFTs should be classified by differential cohomology. But, why is that?

QFTs are expected to form higher categories. Morphisms are given by interfaces, junctions between interfaces...
In order to talk about non-topological QFTs, we should consider smooth variation. QFTs should form a sheaf of higher categories on Mfd.

On the mathematical side, a modern framework of differential cohomology is given in terms of sheaves. I explain the idea to use this framework to the sheaf of invertible QFTs. This is the ongoing joint work with Kantaro Ohmori (U. Tokyo).

General theory : Differential cohomology as sheaves

Bunke, Nikolaus and Völkl, [BNV16] developed a generalized framework of differential cohomology in terms of sheaves on the category of manifolds. Let $\mathcal{C}$ be an $(\infty, 1)$-category. A $\mathcal{C}$-valued sheaf is a contravariant functor

$$\widehat{E} \colon \mathrm{Mfd}^{\mathrm{op}} \to \mathcal{C}$$

with the descent property.

Physically, sheaves describe various fields on manifolds.

- Let N be a manifold. The assignment $X \mapsto C^\infty(X, N)$ is a set-valued sheaf.
- The assignment $\Omega^* \colon X \mapsto \Omega^*(X)$ is a cochain-complex-valued sheaf.
- Let G be a Lie group. The assignment $BG_\nabla \colon X \mapsto$(the groupoid of principal G-bundles with connections over X) is a groupoid-valued sheaf (stack).

Let $\mathcal{C}$ be a stable presentable $(\infty, 1)$-category, such as $\mathrm{Spectra}$, $\mathrm{Ch}[W^{-1}]$. Given a sheaf $\widehat{E} \colon \mathrm{Mfd}^{\mathrm{op}} \to \mathcal{C}$, they associates various sheaves and structure maps.

Evaluating on a manifold X and taking π_{-m}, we get the commutative hexagon

$$
\begin{tikzcd}
 & S(\widehat{E})^m(X) \arrow[rr] \arrow[dr] & & U(\widehat{E})^m(X) \arrow[dr] & \\
Z(\widehat{E})^{m-1}(X) \arrow[ur] \arrow[dr] & & \widehat{E}^m(X) \arrow[ur, "I"] \arrow[dr, "R"] & & Z(\widehat{E})^m(X) \\
 & \mathcal{A}(\widehat{E})^m(X) \arrow[ur, "a"] \arrow[rr, "d"] & & \mathcal{Z}(\widehat{E})^m(X) \arrow[ur] &
\end{tikzcd}
$$

where the diagonal sequences are exact.

$U(\widehat{E}), S(\widehat{E}), Z(\widehat{E}) \in \mathcal{C}$. The interpretations are,

- $U(\widehat{E})$ is the underlying cohomology theory.
- $S(\widehat{E})$ classifies flat classes.
- $Z(\widehat{E})$ classifies differential deformations.

Examples

- The Hopkins-Singer's differential function spectra.
  The imput was a spectrum E, a cochain complex C and $c\colon E \to HC$. It corresponds to the sheaf $\mathrm{Diff}^m(E, C, c)$ with value in $\mathrm{Spectra}$, defined as the pullback

$$\begin{tikzcd}
\mathrm{Diff}^m(E, C, c) \arrow[r] \arrow[d] & H((\Omega \otimes_{\mathbb{R}} C)^{\geq m}) \arrow[d] \\
\mathrm{const}(E) \arrow[r, "\mathrm{const}(c)"] & \mathrm{const}(HC)
\end{tikzcd}.$$

- Let Vect_∇ be a sheaf (stack) of groupoids of hermitian vector bundles with connection. We can construct the corresponding sheaf of spectra $\widehat{ku}_\nabla$. $\widehat{ku}_\nabla$ is a differential refinement of connective K-theory ku.
  We can construct a map $\widehat{ku}_\nabla \to \mathrm{Diff}^0(ku, \mathbb{C}[b], c)$ compatible with the Chern character forms.

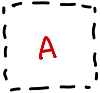
Sheaf of invertible QFTs and differential cohomology

From now on, we only talk about QFTs on stably framed manifolds. (Expected to be classified by $I\mathbb{Z} = I\Omega^{\mathrm{fr}}$.) The basic ansatz is,

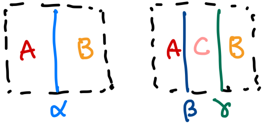
Ansatz 3

There exists a d-category $\widehat{\mathrm{Inv}}_d$ of d-dimensional invertible QFTs and invertible interfaces between them. Moreover, it is a Picard d-groupoid.

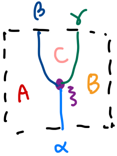
We take it as an ansatz, but lets try to give geometric picture of $\widehat{\mathrm{Inv}}_d$. Intuitively, k-morphisms are given by "diagrams with labelled defects".



object : d-dim invertible QFT



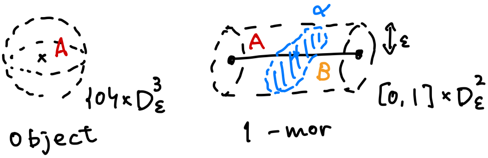
1-morphism : codim 1 invertible interfaces



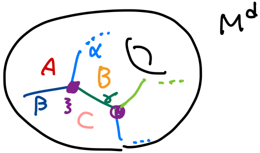
2-morphism : codim 2 invertible junctions

To be more precise,

- Diagrams for k-morphisms should be regarded as germs of $[0,1]^k$ in $\mathbb{R}^d$ and equipped with stable framing.
- Defects should be $\pitchfork$ to $[0,1]^k$ and have stable normal framings.



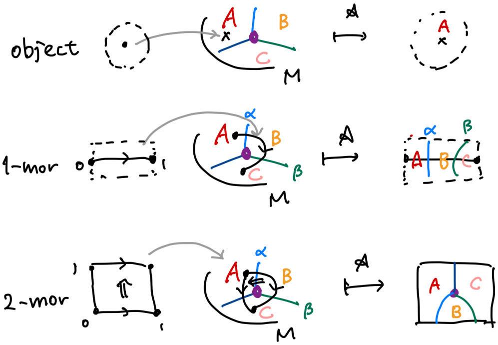
To motivate this, let us take



We claim that this gives a functor from the fundamental ∞-groupoid of M,

$$\mathcal{A}\colon \pi_{\leq\infty} M \to \widehat{\mathrm{Inv}_d}.$$

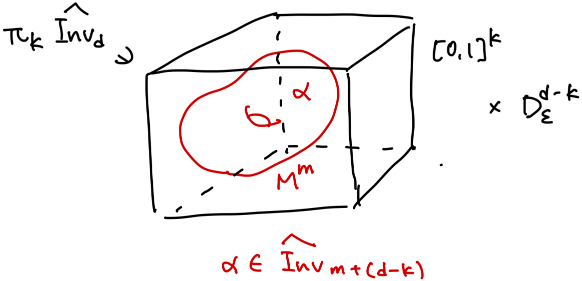
We construct $\mathcal{A}\colon \pi_{\le\infty}M \to \widehat{\mathrm{Inv}}_d$. Indeed, we can read off the defects with labels along the paths in M and homotopies as



Mathematically, $\mathcal{A}\colon \pi_{\le\infty}M \to \widehat{\mathrm{Inv}}_d$ gives an element $[\mathcal{A}] \in (\widehat{\mathrm{Inv}}_d)^0(M)$. The partition function is obtained by its image of differential integration

$$(\widehat{\mathrm{Inv}}_d)^0(M) \to (\widehat{\mathrm{Inv}}_d)^{-d}(\mathrm{pt}) = \mathbb{C}\setminus\{0\}.$$

Another example of k-morphisms in $\widehat{\mathrm{Inv}}_d$:



To encode the smooth variation of QFTs, we need to consider target spaces. We refine Ansatz 3 to the following.

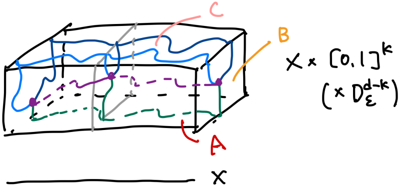
Ansatz 4

For each manifold X, there exists a Picard d-groupoid $\widehat{\mathrm{Inv}}_d(X)$ consisting of d-dimensional invertible QFTs with target X. Moreover, the assignment

$$\widehat{\mathrm{Inv}}_d \colon X \mapsto \widehat{\mathrm{Inv}}_d(X)$$

is a sheaf.

The pictures become bundles over X. An example of k-morphism :



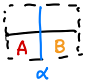
Lets look at examples coming from ordinary differential cocycles.
We can put differential $(k+1)$-cochain on k-dimensional defects.
This gives the functor from the differential cochain complex to $\widehat{\mathrm{Inv}}_d(X)$ by just "putting defects in the middle",

$A \in \hat{Z}^{d+1}(d+1)(X) \longmapsto$

$\times X$

$\alpha \in \hat{C}^{d}(d+1)(X) \longmapsto$

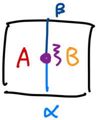
$d\alpha = A - B$



$\times X$

$\xi \in \hat{C}^{d-1}(d+1)(X) \longmapsto$

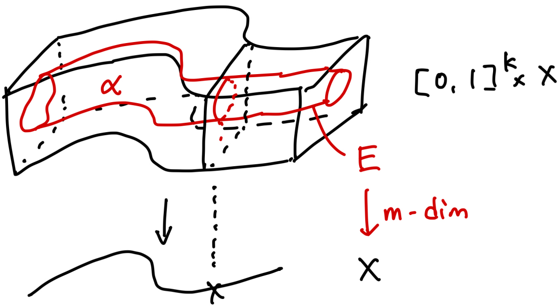
$d\xi = \alpha - \beta$



$\times X$

This refines a transformation $\Sigma^{d+1}H\mathbb{Z} \to \mathrm{Inv}_d$ (expected to be Anderson duality $H\mathbb{Z} \to I\mathbb{Z}$).

Another example of k-morphism :



- $E \to X$ with $\dim E - \dim X = m$, embeded in $[0,1]^k \times X$ with a trivialization of tubular neighborhood.
- $\alpha \in \pi_{k-m}\widehat{\mathrm{Inv}}_d(E)$.

This gives the integration map $\pi_{k-m}\widehat{\mathrm{Inv}}_d(E) \to \pi_k\widehat{\mathrm{Inv}}_d(X)$.

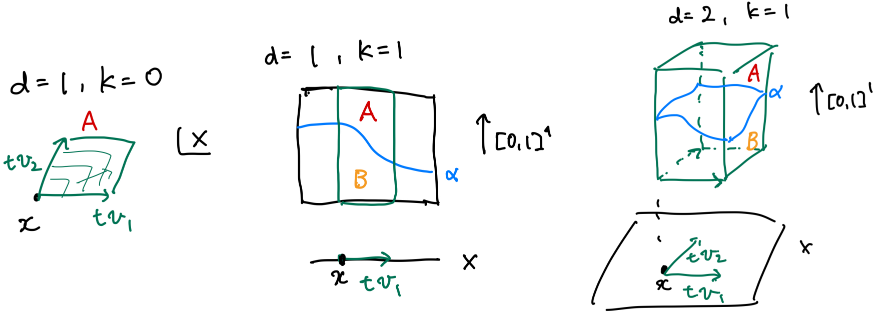
To connect with the differential cohomology in classical sense, we take

Ansatz 5

There exists a morphism of sheaves

$$\omega \colon \widehat{\mathrm{Inv}}_d \to \Omega^{\leq d+1}.$$

ω assigns a $(d+1-k)$-form to a k-morphism.
It is given by taking $\frac{d}{dt}\Big|_{t=0} \arg Z_A$ of the following S^d :



In order to get differential cohomology group, we introduce a filtration on $\widehat{\mathrm{Inv}}_d$ and take homotopy groups.
Once we have such sheaves, we get the hexagon of the form

$$
\begin{tikzcd}
 & * \arrow[rr] \arrow[dr] & & (\mathrm{Inv}_d)^s(X) \arrow[dr] & \\
* \arrow[ur] \arrow[dr] & & (\widehat{\mathrm{Inv}}_d)^s(X) \arrow[ur, "I"] \arrow[dr, "R"] & & * \\
 & * \arrow[ur, "a"] \arrow[rr, "d"] & & * \arrow[ur] &
\end{tikzcd}
$$

with some spectrum Inv_d. (Expected to be $\Sigma^{d+1} I\mathbb{Z}$.)
Under Ansatz 5, by the abstract nonsense we get a transformation

$$\widehat{\mathrm{Inv}}_d^s \to \mathrm{Diff}^{d+1-s}(\mathrm{Inv}_d, \mathbb{R}[d+1], c)$$

to the sheaf of differential function spectra, representing differential Inv_d in the classical sense.

Questions

- One good point for formulating differential cohomology as sheaves is that we can treat various structures, such as multiplicative structures and twisted differential cohomology and other higher operations, systematically.
  Can we describe their relations with physics using our viewpoint?
- Dualy to the differential cohomology, differential *homology* should be formulated in terms of cosheaves. What is the relations with factorization homology?
  The construction $\widehat{E}_* \mapsto \widehat{IE}^*$ should be enhanced to the construction (differential homology cosheaf) $\mapsto$ (Anderson dual sheaf).
  Our $\widehat{\mathrm{Inv}_d}$ should be the special case of this.

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