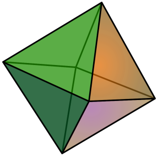
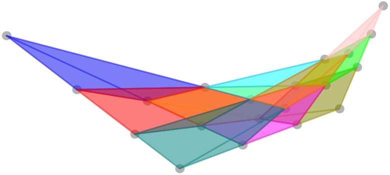
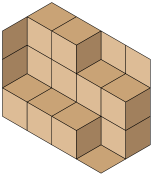
The positive Grassmannian, the amplituhedron, and cluster algebras

Lauren K. Williams, Harvard

Based on: joint works with Steven Karp, Tomasz Lukowski, Matteo Parisi, Melissa Sherman-Bennett, Yan X. Zhang, ...

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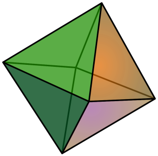
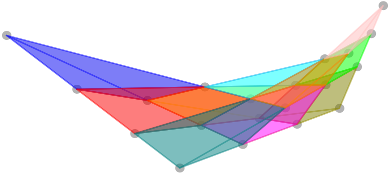
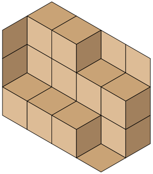
Overview of the talk

- Background on the positive Grassmannian
- Background and motivation for the amplituhedron ($\mathcal{N} = 4$ SYM)
- The amplituhedron is connected to:
    - tilings/triangulations
    - cluster algebras
    - the positive tropical Grassmannian
    - plane partitions

The Grassmannian and the matroid stratification

The **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$
Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix C.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plücker coordinate** $p_I(C)$ is the minor of the $k \times k$ submatrix of C in column set I.

The *matroid* associated to $C \in Gr_{k,n}$ is $\mathcal{M}(C) := \{I \in \binom{[n]}{k} \mid p_I(C) \neq 0.\}$

Gelfand-Goresky-MacPherson-Serganova '87 introduced the *matroid stratification* of $Gr_{k,n}$.

Given $\mathcal{M} \subset \binom{[n]}{k}$, let $\mathrm{S}_{\mathcal{M}} = \{C \in Gr_{k,n} \mid p_I(C) \neq 0 \text{ iff } I \in \mathcal{M}\}$.

$$\text{Matroid stratification: } Gr_{k,n} = \sqcup_{\mathcal{M}} \mathrm{S}_{\mathcal{M}}.$$

However, the topology of matroid strata is terrible –
Mnev's *universality theorem* (1987).

What is the positive Grassmannian?

Background: 1994 Lusztig total positivity for G/P, 1997 Rietsch, 2006 Postnikov preprint on *totally non-negative* (TNN) or "positive" Grassmannian.

Let $Gr_{k,n}^{\geq 0}$ be subset of $Gr_{k,n}(\mathbb{R})$ where Plucker coords $p_I \geq 0$ for all I.

Inspired by matroid stratification, one can partition $Gr_{k,n}^{\geq 0}$ into pieces based on which Plücker coordinates are positive and which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}} := \{C \in Gr_{k,n}^{\geq 0} \mid p_I(C) > 0 \text{ iff } I \in \mathcal{M}\}$.

In contrast to terrible topology of matroid strata ...

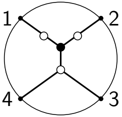
(Postnikov, see also Rietsch) If $S_{\mathcal{M}}$ is non-empty it is a (positroid) *cell*, i.e. homeomorphic to an open ball. So we have *positroid cell decomposition*

$$Gr_{k,n}^{\geq 0} = \sqcup S_{\mathcal{M}}.$$

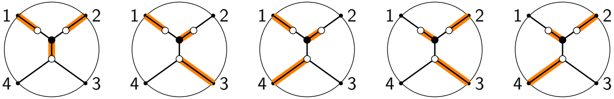
Can classify the nonempty cells ...

How to read off a positroid cell from a plabic graph

- Positroid cells $\leftrightarrow$ *plabic graphs*, planar graphs embedded in disk with boundary vertices labeled $1, 2, \ldots, n$ and internal vertices colored black or white.



- WLOG we assume graph G is bipartite and that every boundary vertex is incident to a white vertex.
- Let $\mathcal{M}(G) := \{\partial(P) \mid P$ is an **almost perfect matching** of $G\}$.



E.g. for graph above, get $\mathcal{M}(G) = \{12, 13, 14, 23, 24\}$.

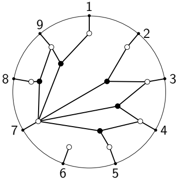
- Theorem (Postnikov): $\mathcal{M}(G)$ is the set of nonzero Plücker coordinates of a positroid cell, and all cells obtained this way.

Classification of positroid cells

Theorem (Postnikov): The positroid cells of $Gr_{k,n}^{\geq 0}$ are in bijection with:

- equivalence classes of *plabic graphs* (*on-shell graphs*)
- *decorated permutations* π on $[n]$ with k antiexcedances

So we'll refer to cells as S_π.



Associate dec. permutation to (reduced) plabic graph by *zig-zag paths*:

- From each boundary vertex i, turn right at black, left at white, to reach some other boundary vertex j. Then set $\pi(i) := j$.
- If have a *white/black lollipop* at i, set $\pi(i) = \bar{i}$ or $\pi(i) = \underline{i}$.

Here $\pi = (8, 4, 5, 7, 2, \bar{6}, 9, 1, 3)$.

Where did the amplituhedron come from?

Motivation for the amplituhedron ($\mathcal{N} = 4$ SYM):

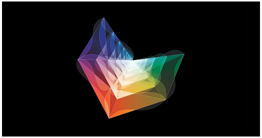
- the recurrence of Britto–Cachazo–Feng–Witten (2005) expresses scattering amplitudes as sums of rat'l functions of momenta. Indiv terms have "spurious poles" – singularities not present in amplitude.
- Hodges (2009) observed that in some cases, the amplitude is the volume of a polytope, with spurious poles arising from internal boundaries of a triangulation of the polytope. Asked if in general each amplitude is the volume of some geometric object.
- AH–T found the amplituhedron as the answer to this question; BCFW recurrence is interpreted as "triangulation" of $\mathcal{A}_{n,k,4}(Z)$.

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$, Arkani-Hamed–Trnka (2013).

Fix n, k, m with $k + m \leq n$.
Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ be an $n \times (k+m)$ matrix with max'l minors positive.
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to span(CZ).
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Amplituhedron in the press

- A "jewel at the heart of quantum physics" – Wired Magazine.



- #10 among the 100 top stories of 2013, Discover Magazine.

- One of the 25 best inventions of the year 2013, Time Magazine.

  "The new method represents probabilities as pyramid-like structures, then combines the pyramids into one elegant gemstone-like structure called an amplituhedron,. . . "

Physics motivation for the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}^{+}_{n,k+m}$ (max minors > 0).
Let $\widetilde{Z}$ be map $(Gr_{k,n})_{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix A to AZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}((Gr_{k,n})_{\geq 0}) \subset Gr_{k,k+m}$.

Amplituhedron makes sense for any m. Special cases:

- The $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}$:
  - encodes the geometry of (tree-level) scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- The $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$:
  - considered a toy-model for $m = 4$ case.
  - governs geometry of scattering amplitudes in $\mathcal{N} = 4$ SYM at subleading order in perturbation theory for the 'MHV' sector of the theory *(cf def of loop amplituhedron)*.
  - is relevant to the 'next to MHV' sector, enhancing connection with geometries of loop amplitudes (Kojima–Langer).

Examples of the amplituhedron

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- If $m = n - k$, $\mathcal{A}_{n,k,m}(Z) = Gr_{k,n}^{\geq 0}$.

Examples of the amplituhedron

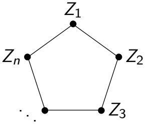
The amplituhedron $\mathcal{A}_{n,k,m}(Z)$

Fix n, k, m with $k+m \le n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ (max minors >0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

Special cases:

- If $k=1$ and $m=2$, $\mathcal{A}_{n,k,m} \subset Gr_{1,3}$ is equivalent to an n-gon in $\mathbb{RP}^2$:

  Write $Z = \begin{pmatrix} Z_1 \\ \hline \vdots \\ \hline Z_n \end{pmatrix} \in \mathrm{Mat}_{n,3}^{>0}$.

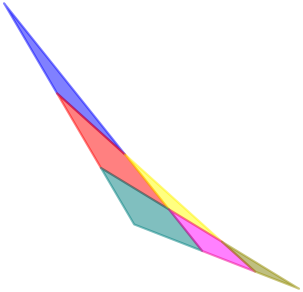


  Pos minors $\Rightarrow Z_1, \dots, Z_n$ correspond to vertices of polygon in $\mathbb{RP}^2$.
  Map $\tilde{Z} : \mathrm{Gr}_{1,n}^{\geq 0} \to \mathrm{Gr}_{1,3}$ sends $(c_1, \dots, c_n) \mapsto c_1 Z_1 + \cdots + c_n Z_n$.
  As c_i's range over $\mathbb{R}^{\geq 0}$ we get all points of polygon.
- For $k=1$ and general m, get cyclic polytope in $\mathbb{RP}^m$.

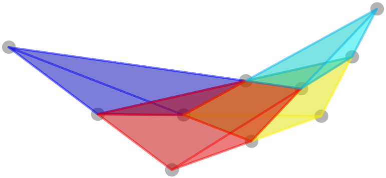
Examples of the amplituhedron

Fix n, k, m with $k + m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. Let $\widetilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending $C \mapsto CZ$. Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

- If $m = 1$, $\mathcal{A}_{n,k,m}(Z) \subset Gr_{k,k+1}$ is homeomorphic to the bounded complex of the cyclic hyperplane arrangement of n hyperplanes in $\mathbb{R}^k$ defined by Z (Karp–W.)



$\mathcal{A}_{5,2,1}$



$\mathcal{A}_{5,3,1}$

How can we understand the amplituhedron?

Fix n, k, m with $k+m \leq n$, let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. Let $\widetilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ map $C \mapsto CZ$. Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$. $\dim \mathcal{A}_{n,k,m} = km$.

We understand $\mathrm{Gr}_{k,n}^{\geq 0} = \sqcup S_\pi$ quite well, and $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \twoheadrightarrow \mathcal{A}_{n,k,m}(Z)$, so we want to understand $\mathcal{A}_{n,k,m}(Z)$ using images $\tilde{Z}(S_\pi)$ of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$.

Some questions:

- When is $\tilde{Z}$ injective on a cell S_π?
  (If $\dim S_\pi = km$, and $\tilde{Z}|_{S_\pi}$ injective, call $\overline{\tilde{Z}(S_\pi)}$ a *tile* of $\mathcal{A}_{n,k,m}(Z)$.)
- Can one find collection of tiles which fit together to "tile" $\mathcal{A}_{n,k,m}(Z)$?
- How many tiles comprise a tiling? Can we understand all tilings?
- Can we describe $\mathcal{A}_{n,k,m}(Z)$ directly inside $\mathrm{Gr}_{k,k+m}$?
- Is there a connection to cluster algebras?

How can we understand the amplituhedron?

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  (If $\dim S_\pi = km$, and $\tilde{Z}|_{S_\pi}$ injective, call $\tilde{Z}(S_\pi)$ a *tile* of $\mathcal{A}_{n,k,m}(Z)$.)
- When does a collection of tiles fit together to "tile" $\mathcal{A}_{n,k,m}(Z)$?
- How many tiles comprise a tiling? Can we understand all tilings?
- Can we describe $\mathcal{A}_{n,k,m}(Z)$ directly inside $\mathrm{Gr}_{k,k+m}$?
- Is there a connection to cluster algebras?

Have good understanding when

- $k = 1$ (cyclic polytope case),
- $m = 1$ (bounded complex of cyclic hyperplane arrangement, Karp-W).

Recently we've also understood most of these questions for $m = 2$ (Lukowski–Parisi–W, Parisi–Sherman-Bennett–W).

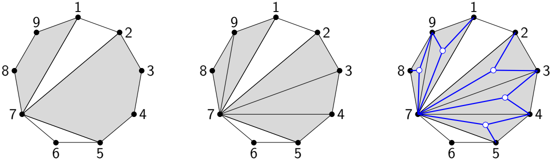
Tiles of the amplituhedron

Recall: $\overline{\tilde{Z}(S_\pi)}$ is a *positroid tile* for $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ if $\tilde{Z}$ is injective on km-dim'l cell S_π. Lukowski–Parisi–Spradlin–Volovich gave a conjectural characterization of tiles.

Theorem (Parisi–Sherman-Bennett–W)

The positroid tiles for $\mathcal{A}_{n,k,2}(Z) \leftrightarrow$ collections of nonoverlapping grey polygons in an n-gon with total "area" k. To construct the cell S_π:

- Choose triangulation of grey polygons into k grey triangles.
- Put white vertex in every grey triangle, connected to three vertices.
- Elements of S_π are the $k \times n$ *Kasteleyn matrices* with rows/columns indexed by the white and black vertices.



$$
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
* & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
0 & * & * & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & * & * & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & * & * & 0 & * & 0 & 0
\end{bmatrix}
\end{array}
$$

Tiles and cluster algebras

Cluster algebras (Fomin–Zelevinsky)

Cluster algebras are a class of commutative rings with remarkable combinatorial structure. They come with distinguished generators called *cluster variables*, and relations are encoded by *quivers* and *quiver mutation*.

Cluster varieties are varieties whose coordinate rings are cluster algebras; they come with many nice torus charts.

Examples: Grassmannians, flag varieties, Schubert varieties, ...

- It's useful to exhibit a cluster structure because of the many general results about them (Laurent phenomenon, positivity theorem, etc).
- Is there a cluster structure for the amplituhedron?

Clue that answer should be yes:
Physicists had observed that when one calculates scattering amplitudes as rat'l functions of momenta, the poles arising in expressions seemed to be related to compatible collections of cluster variables ("cluster adjacency").

Coordinates for the amplituhedron

Fix n, k, m with $k+m \leq n$, let $Z \in \text{Mat}_{n,k+m}^{>0}$ (max minors > 0).
Let $\widetilde{Z}$ be map $Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending a $k \times n$ matrix C to CZ.
Set $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$.

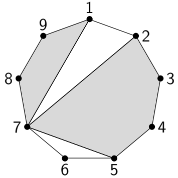
Need coordinates so we can describe $\mathcal{A}_{n,k,m}$ directly inside $Gr_{k,k+m}$.
Let $Z_1, \dots, Z_n$ be rows of Z. Let $Y \in Gr_{k,k+m}$ (viewed as matrix).
Given $I = \{i_1 < \cdots < i_m\} \subset [n]$, let

$$\langle YZ_I \rangle = \langle YZ_{i_1} \dots Z_{i_m} \rangle := \det \begin{bmatrix} - & Y & - \\ - & Z_{i_1} & - \\ & \vdots & \\ - & Z_{i_m} & - \end{bmatrix}$$

Call it *twistor coordinate* $\langle YZ_I \rangle$ (Arkani-Hamed–Thomas–Trnka).
Rk: $Y \in Gr_{k,k+m}$ determined by twistor coords; $\langle YZ_I \rangle = p_I(Y^\perp Z^t)$.

Tiles and cluster algebras

Recall: $\overline{\tilde{Z}(S_\pi)}$ is a *positroid tile* for $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ if $\tilde{Z}$ is injective on km-dimensional cell S_π.
Tiles for $\mathcal{A}_{n,k,2}(Z)$ in bijection with collections of nonoverlapping grey polygons in an n-gon with total "area" k.



Theorem (Parisi–Sherman-Bennett–W)

Get cluster structure on each tile for $m = 2$. Cluster/frozen variables have form $\pm\langle YZ_aZ_b\rangle$, where ab is a diagonal or edge of a grey polygon.

Can also use twistor coordinates to provide inequality description of tiles, and a concrete description of $\mathcal{A}_{n,k,2}(Z)$ as subset of $\mathrm{Gr}_{k,k+2}$ (P-SB-W), proving a conjecture of Arkani-Hamed–Thomas–Trnka.

Tilings of the amplituhedron

Recall: for $Z \in \mathrm{Mat}_{n,k+m}^{>0}$, have $\widetilde{Z} : Gr_{k,n}^{\geq 0} \to Gr_{k,k+m}$ sending $C \mapsto CZ$.
Amplituhedron $\mathcal{A}_{n,k,m}(Z) := \widetilde{Z}(Gr_{k,n}^{\geq 0}) \subset Gr_{k,k+m}$ has dim km.

- We've studied *(positroid) tiles*, i.e. images $\overline{\tilde{Z}(S_\pi)}$ of km-dim'l cells S_π where $\tilde{Z}$ injective. Can we fit some tiles together to "tile" or "triangulate" the amplituhedron?

A $\tilde{Z}$*-induced tiling* (or positroid tiling) of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{\overline{\tilde{Z}(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of km-dimensional cells, such that:

- $\tilde{Z}$ is injective on each S_π for $\pi \in \mathcal{C}$ ($\overline{\tilde{Z}(S_\pi)}$ a *tile*)
- their union equals $\mathcal{A}_{n,k,m}(Z)$
- their interiors are pairwise disjoint

When $m = 4$, Arkani-Hamed–Trnka conjectured that certain "BCFW cells" give a tiling of $\mathcal{A}_{n,k,4}(Z)$; proved by Even-Zohar–Lakrec–Tessler.

Tilings of the amplituhedron

Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,m}(Z)$ a continuous surjective map onto km-dim'l amplituhedron $\mathcal{A}_{n,k,m}(Z)$.

A *$\tilde{Z}$-induced tiling* (or positroid tiling) of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{\overline{\tilde{Z}(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of km-dimensional cells, such that:

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- their union equals $\mathcal{A}_{n,k,m}(Z)$
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- Can we understand when tiles for the amplituhedron fit together to form a tiling? Can we classify all the tilings?
- Seems very hard in general, but for $m = 2$ we can classify them.
- Surprisingly, this story is related to the *moment map* for the Grassmannian and the *positive tropical Grassmannian* (Lukowski–Parisi–W, Parisi–Sherman-Bennett–W).

The moment map for the Grassmannian

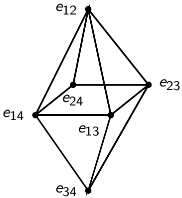
Let $\{e_1, \dots, e_n\}$ be basis of $\mathbb{R}^n$; for $I \subset [n]$, let $e_I := \sum_{i \in I} e_i \in \mathbb{R}^n$.

The **moment map** $\mu : \mathrm{Gr}_{k,n} \to \mathbb{R}^n$ (for the T^n action on $Gr_{k,n}(\mathbb{C})$) is defined by

$$\mu(C) = \frac{\sum_{I \in \binom{[n]}{k}} |p_I(C)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(C)|^2} \in \mathbb{R}^n.$$

The moment map image $\mu(\mathrm{Gr}_{k,n})$ is an $(n-1)$-dim'l polytope in $\mathbb{R}^n$ called the *hypersimplex* $\Delta_{k,n} = \mathrm{Conv}\{e_I \ : \ I \in \binom{[n]}{k}\}$.

$\Delta_{2,4}$:



The moment map for the Grassmannian

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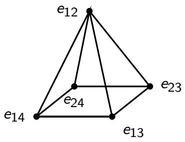
The **moment map** $\mu : \mathrm{Gr}_{k,n} \to \mathbb{R}^n$ is $\quad \mu(C) = \dfrac{\sum_{I \in \binom{[n]}{k}} |p_I(C)|^2 e_I}{\sum_{I \in \binom{[n]}{k}} |p_I(C)|^2} \in \mathbb{R}^n.$

- We can associate two objects to matroid $\mathcal{M} \subset \binom{[n]}{k}$
  - *matroid stratum* $S_{\mathcal{M}} := \{C \in Gr_{k,n} \mid p_I(C) \neq 0 \text{ iff } I \in \mathcal{M}\}$.
  - *matroid polytope* $\Gamma_{\mathcal{M}} = \mathrm{Conv}\{e_I \mid I \in \mathcal{M}\}$.
- Gelfand–Goresky–MacPherson–Serganova used the Convexity Theorem to show that for $S_{\mathcal{M}} \neq \emptyset$, we have $\overline{\mu(S_{\mathcal{M}})} = \Gamma_{\mathcal{M}}$.

$\mathcal{M} = \{12, 13, 14, 23, 24\} \rightsquigarrow$



- What does all this have to do with the amplituhedron?

Recap on hypersimplex and $m = 2$ amplituhedron

- Hypersimplex $\Delta_{k+1,n}$ is an $(n-1)$-dimensional polytope in $\mathbb{R}^n$.
- Is the image of $\mathrm{Gr}_{k+1,n}^{\geq 0}$ under the moment map μ.
- Can define μ-induced tilings of $\Delta_{k+1,n}$.
- Tiles will be certain $(n-1)$-dimensional matroid polytopes in $\mathbb{R}^n$.

Meanwhile,

- $\mathcal{A}_{n,k,2}(Z)$ is a non-polytopal $2k$-dimensional subset of $\mathrm{Gr}_{k,k+2}$.
- Is the image of $\mathrm{Gr}_{k,n}^{\geq 0}$ under amplituhedron map $\tilde{Z}$.
- Can consider $\tilde{Z}$-induced tilings of $\mathrm{Gr}_{k,k+2}$.
- Tiles will be certain $2k$-dimensional subsets of $\mathrm{Gr}_{k,k+2}$.

Remarkably, tilings of hypersimplex $\Delta_{k+1,n}$ and of $\mathcal{A}_{n,k,2}(Z)$ are related!

Tilings of the amplituhedron

- Have $Gr_{k+1,n}^{\geq 0} = \sqcup_\pi S_\pi$, and moment map $\mu : Gr_{k+1,n}^{\geq 0} \to \Delta_{k+1,n} \subset \mathbb{R}^n$.
  A *μ-induced tiling* of $\Delta_{k+1,n}$ is a collection $\{\overline{\mu(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of $(n-1)$-dimensional cells, such that:
  μ injective on each S_π; union equals $\Delta_{k+1,n}$; interiors pairwise disjoint.
- Have $Gr_{k,n}^{\geq 0} = \sqcup_\pi S_\pi$, and amp map $\tilde{Z} : Gr_{k,n}^{\geq 0} \to \mathcal{A}_{n,k,2}(Z) \subset \mathrm{Gr}_{k,k+2}$.
  A *$\tilde{Z}$-induced tiling* of $\mathcal{A}_{n,k,2}(Z)$ is a collection $\{\overline{\tilde{Z}(S_\pi)} \mid \pi \in \mathcal{C}\}$ of closures of images of $2k$-dimensional cells, such that:
  $\tilde{Z}$ injective on each S_π; union equals $\mathcal{A}_{n,k,2}(Z)$; interiors pairwise disjoint.

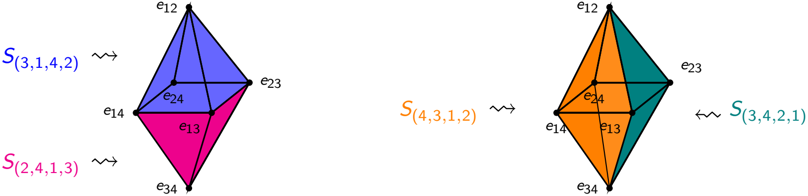
Conj (Lukowski-Parisi-W); Thm (Parisi–Sherman-Bennett–W.)

There is a bijection (*T-duality*)
$\{$(Loopless) cells S_π of $\mathrm{Gr}_{k+1,n}^{\geq 0}\} \to \{$(coloopless) cells $S_{\hat{\pi}}$ of $\mathrm{Gr}_{k,n}^{\geq 0}\}$ s.t.

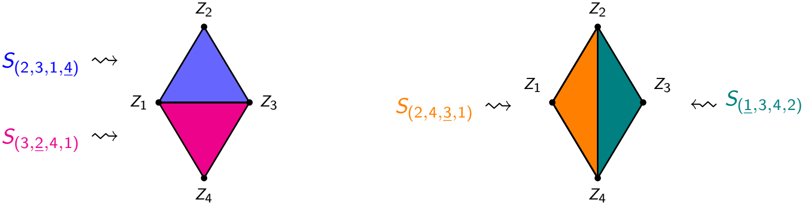
- μ injective on $(n-1)$-dim'l cell S_π iff $\tilde{Z}$ injective on $2k$-dim'l cell $S_{\hat{\pi}}$.
- A collection $\{\overline{\mu(S_\pi)}\}$ is tiling of $\Delta_{k+1,n}$ iff $\{\overline{\tilde{Z}(S_{\hat{\pi}})}\}$ is tiling of $\mathcal{A}_{n,k,2}(Z)$ for all $Z \in \mathrm{Mat}_{n,k+2}^{>0}$.

Example of theorem (when $k = 1$ and $n = 4$)

$\mu : \mathrm{Gr}_{2,4}^{\geq 0} \to \Delta_{2,4} \rightsquigarrow$ two tilings of hypersimplex $\Delta_{2,4}$ (octahedron):



$\tilde{Z} : \mathrm{Gr}_{1,4}^{\geq 0} \to \mathcal{A}_{4,1,2}(Z) \rightsquigarrow$ two tilings of $\mathcal{A}_{4,1,2}(Z)$ (a quadrilateral in $\mathbb{P}^2$):

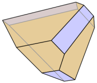


How can we biject the cells of these tilings?

T-duality map $\pi = (a_1, a_2, \ldots, a_n) \mapsto \hat{\pi} := (a_n, a_1, a_2, \ldots, a_{n-1})$.

Connection to the positive tropical Grassmannian

- The *positive tropical Grassmannian* $\mathrm{Trop}^+ \mathrm{Gr}_{k,n}$ is a tropical analogue of the positive Grassmannian. Can be defined by tropicalizing the



  Plücker relations (Speyer-W).

- Points $P = \{P_I\}_I \in \mathbb{R}^{\binom{[n]}{k}}$ in $\mathrm{Trop}^+ \mathrm{Gr}_{k,n}$ – interpreted as height functions on $\Delta_{k,n}$ – give rise to regular positroid subdivisions of the hypersimplex $\Delta_{k,n}$ (Lukowski–Parisi–W, Arkani-Hamed–Lam–Spradlin).

- Recall: have a bijection between μ-induced tilings of the hypersimplex $\Delta_{k+1,n}$ and $\tilde{Z}$-induced tilings of the amplituhedron $\mathcal{A}_{n,k,2}$.

- Corollary (Parisi–Sherman-Bennett–W): Points in (maximal cones of) $\mathrm{Trop}^+ \mathrm{Gr}_{k+1,n}$ give rise to tilings of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$.

- Is there an analogue of the positive tropical Grassmannian which controls tilings of the amplituhedron for $m \neq 2$?

Tilings of the amplituhedron

$\tilde{Z}$-induced tilings have been studied in special cases. Their cardinalities are interesting!

special case	cardinality of tiling of $\mathcal{A}_{n,k,m}(Z)$	explanation
$m = 0$ or $k = 0$	1	$\mathcal{A}$ is a point
$k + m = n$	1	$\mathcal{A} \cong \mathrm{Gr}_{k,n}^{\geq 0}$
$m = 1$	$\binom{n-1}{k}$	Karp-W.
$m = 2$	$\binom{n-2}{k}$	AH-T-T, Bao-He, P-SB-W
$m = 4$	$\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$	AH-T, Even-Zohar–Lakrec–Tessler
$k = 1$, m even	$\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$	$\mathcal{A} \cong$ cyclic polytope $C(n,m)$

Tilings of the amplituhedron

Wild conjecture (Karp-Zhang-W)

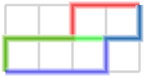
$$\text{Let } M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

A $\tilde{Z}$-induced tiling of $\mathcal{A}_{n,k,m}$ for even m has cardinality $M(k, n-k-m, \frac{m}{2})$.

Remark: Consistent with results for $m=2, m=4, k=1$. Symmetries!

The number $M(a,b,c)$ counts: (In figure, $a, b, c = 2, 4, 3$.)

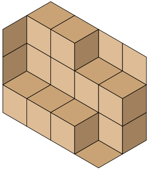
noncrossing lattice paths



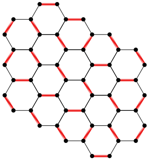
plane partition

3	3	2	2
1	1	1	

rhombic tiling



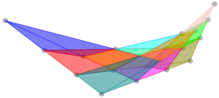
perfect matching



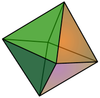
Summary

The amplituhedron includes as special cases

- the positive Grassmannian
- cyclic polytopes
- bounded complex of cyclic hyperplane arrangement



and is closely connected to the hypersimplex.

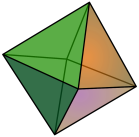
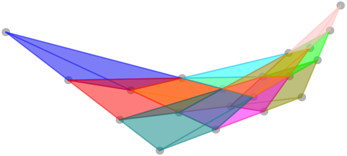
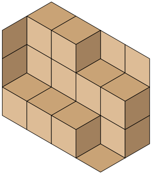


It is useful to study amplituhedron from the point of view of

- matroids,
- cluster algebras,
- tilings.
- tropical geometry.

Many open questions!

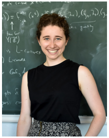
Thank you for listening!

- "The $m = 1$ amplituhedron and cyclic hyperplane arrangements," with Steven Karp, arXiv:1608.08288
- "Decompositions of amplituhedron," with Karp and Yan X. Zhang, arXiv:1708.09525
- "The positive tropical Grassmannian, the hypersimplex, and the $m = 2$ amplituhedron," with Tomasz Lukowski and Matteo Parisi, arXiv:2002.06164
- "The $m = 2$ amplituhedron and the hypersimplex: signs, clusters, triangulations, Eulerian numbers," with Parisi and Melissa Sherman-Bennett, arXiv:2104.08254.

- "The positive Grassmannian, the amplituhedron, and cluster algebras," to appear in the Proceedings of the 2022 ICM.




Remarks on T-duality

- $\mathcal{N} = 4$ SYM has a string theory description and one can study the T-dual theory of this string theory. In particular, *scattering amplitudes* are T-dual to *Wilson loops* (Berkovits–Maldacena).
- Wilson loops can be calculated using the amplituhedron $\mathcal{A}_{n,k,4}$ and scattering amplitudes can be calculated using the (related) *momentum amplituhedron* $\mathcal{M}_{n,k,4}$ (defined by Damgaard–Ferro–Lukowski–Parisi). A certain *T-duality map* on permutations[1] conjecturally translates between these two sides.
- When $m = 2$, our analogous *T-duality map* connects the geometry of the amplituhedron $\mathcal{A}_{n,k,2}$ to the hypersimplex $\Delta_{k+1,n}$ (Lukowski–Parisi–W, Parisi–Sherman-Bennett-W).

[1] defined by Arkani-Hamed–Bourjaily–Cachazo–Goncharov–Postnikov–Trnka