Liouville CFT
from probabilistic construction to bootstrap solution

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Based on numerous works with G. Baverez, F. David, C. Guillarmou, A. Kupiainen and R. Rhodes

Context

Statistical physics model in 2D at criticality

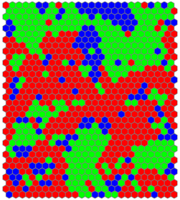
Conformal Field Theories

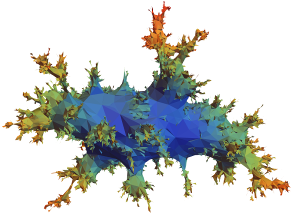
Belavin-Polyakov-Zamolodchikov (1984): Conformal Bootstrap

Maths axioms: Borcherds-Frenkel, Gawedzki, Kontsevich, Segal,...

Context

Discrete statistical physics models $\quad F \mapsto \sum_{\sigma \text{ config}} F(\sigma) e^{-H_\beta(\sigma)}$



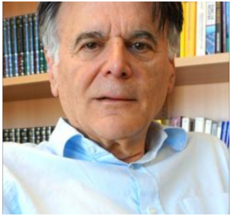


Continuum statistical physics models $\quad F \mapsto \int F(\Phi) e^{-S(\Phi)} \, D\Phi =: \langle F \rangle$

Fields $x \in \Sigma \mapsto V_\alpha(\Phi, x) := V_\alpha(x)$ indexed by α

Correlation functions:

$$\langle V_{\alpha_1}(x_1) \dots V_{\alpha_m}(x_m) \rangle$$



" The manuscript that follows was written fifteen years ago...I just wanted to justify my proposed definition by checking that all know examples of CFTs did fit the definition. This task held me up...."

GRAEME SEGAL, *The definition of Conformal Field Theory* (2004).

Our work: Path integral and Conformal Bootstrap for Liouville CFT via Segal

Path integral for Liouville CFT (Polyakov 81')

Riemann surface Σ, Riemannian metric g

$$F \mapsto \int F(\Phi) e^{-S_\Sigma(\Phi,g)} D\Phi$$

Liouville action

$$S_\Sigma(\Phi, g) = \frac{1}{4\pi} \int_\Sigma (|d\Phi|_g^2 + QK_g\Phi + \mu e^{\gamma\Phi}) \mathrm{dv}_g$$

Parameters

$$\underline{\gamma \in (0,2)}, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2} \quad , \mu > 0$$

Gaussian Free Field

Let X_g be the GFF on Σ in the metric g on Σ

$$X_g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\geq 1} \frac{\alpha_n}{\sqrt{\lambda_n}} e_n(x)$$

with

- $(\alpha_n)_n$ iid standard Gaussians
- $(e_n)_n$ orthonormal basis of eigenfunctions of Laplacian Δ_g with eigenvalues $(\lambda_n)_n$ and b.c. $\int_\Sigma e_n \mathrm{dv}_g = 0$

Gaussian integral:

$$\boxed{\int F(\Phi) e^{-\frac{1}{4\pi}\int_\Sigma |d\Phi|_g^2 \mathrm{dv}_g} D\Phi = (\det{}'(\Delta_g)/\mathrm{v}_g(\Sigma))^{-1/2} \int_{\mathbb{R}} \mathbb{E}\Big[F(c+X_g)\Big]\, dc}$$

Liouville path integral (DGKRV 14-16')

$$\langle F\rangle_{\Sigma,g} := (\det'(\Delta_g)/\mathrm{v}_g(\Sigma))^{-1/2}\int_{\mathbb{R}}\mathbb{E}\Big[F(c+X_g)e^{-\frac{1}{4\pi}\int_\Sigma (QK_g(c+X_g)+\mu e^{\gamma(c+X_g)})\mathrm{dv}_g}\Big]\,dc$$

where

- $\mu > 0$, $\gamma \in (0,2)$ and $Q = 2/\gamma + \gamma/2$
- X_g be the GFF on Σ in the metric g
- $e^{\gamma X_g}\mathrm{dv}_g$ is a random measure (Gaussian multiplicative chaos, Kahane '85)

$$e^{\gamma X_g(x)}\mathrm{dv}_g(x) := \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{g,\epsilon}(x)}\mathrm{dv}_g(x)$$

with $X_{g,\epsilon}$ a regularization of X_g

Correlation functions

Set for $\alpha \in \mathbb{R}$ and $x \in \Sigma$

$$V_\alpha(x) := e^{\alpha\Phi(x)}$$

Correlation functions

$$\langle \prod_{j=1}^{m} V_{\alpha_j}(x_j) \rangle_{\Sigma,g}$$

Theorem (DGKRV '14 -'16): The correlation functions are non trivial iff the Seiberg bounds are satisfied

$$\forall j \quad \alpha_j < Q \qquad \text{and} \qquad \sum_{j=1}^{m} \alpha_j > \chi(\Sigma) Q$$

with $\chi(\Sigma)$ the Euler characteristics.

An explicit expression for the correlation functions in terms of moments of chaos

$$\langle\prod_{k=1}^{n} V_{\alpha_k}(x_k)\rangle_{\Sigma,g} = B \times \mathbb{E}\Big[\mathcal{G}^{\gamma}_{g,(x_k)_k,\alpha}(\Sigma)^{-\frac{\sum_{k=1}^n \alpha_k - \chi(\Sigma)Q}{\gamma}}\Big]$$

where

$$\mathcal{G}^{\gamma}_{g,(x_k)_k,\alpha}(dx) := e^{\gamma \sum_{k=1}^n \alpha_k G_g(x_k,x)} e^{\gamma X_g(x)} dv_g(x)$$

B is explicit:

$$B = \Big(\frac{\det'(\Delta_g)}{v_g(\Sigma)}\Big)^{-\frac{1}{2}} e^{C((x_k)_k)} \mu^{-\sum_{k=1}^n \alpha_k + \chi(\Sigma)Q} \Gamma\Big(\sum_{k=1}^{n} \alpha_k - \chi(\Sigma)Q\Big)$$

where

$$C((x_k)_k) := \sum_{k=1}^{n} \frac{\alpha_k^2}{2} W_g(x_k) + 2\sum_{i<j} \alpha_i \alpha_j G_g(x_i, x_j).$$

for some function W

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Geometric rules

- **Diffeomorphism invariance:** for $\psi : \Sigma \to \Sigma$ diffeo

$$\langle \prod_j V_{\alpha_i}(\psi(x_j)) \rangle_{\Sigma,g} = \langle \prod_j V_{\alpha_j}(x_j) \rangle_{\Sigma,\psi^* g}$$

- **Local scale invariance:** for $\varphi : \Sigma \to \mathbb{R}$ smooth

$$\langle \prod_i V_{\alpha_i}(z_i) \rangle_{\Sigma,e^{\varphi} g} = e^{\frac{\mathbf{c}_L}{96\pi} \int_\Sigma |d\varphi|_g^2 + 2K_g \varphi} \Big(\prod_i e^{-\Delta_{\alpha_i} \varphi(z_i)} \Big) \langle \prod_i V_{\alpha_i}(z_i) \rangle_{\Sigma,g}$$

where

- $\mathbf{c}_L = 1 + 6Q^2$ is the central charge
- conformal weight of the primary field V_α

$$\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$$

Riemann sphere $\hat{\mathbb{C}}$ and structure constants

3 point correlation function (g_0 round metric on sphere)

$$\langle V_{\alpha_1}(x_1) V_{\alpha_2}(x_2) V_{\alpha_3}(x_3) \rangle_{\hat{\mathbb{C}}, g_0}$$

Structure constant

$$C(\alpha_1, \alpha_2, \alpha_3) := \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}, g_0}$$

DOZZ formula (KRV '17)

$$C(\alpha_1,\alpha_2,\alpha_3) = \Big(\pi\,\mu\,\ell(\frac{\gamma^2}{4})\,(\frac{\gamma}{2})^{2-\gamma^2/2}\Big)^{\frac{2Q-\bar\alpha}{\gamma}} \times \frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar\alpha-2Q}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar\alpha-2\alpha_1}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar\alpha-2\alpha_2}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar\alpha-2\alpha_3}{2})}$$

with

$$\bar\alpha = \alpha_1+\alpha_2+\alpha_3, \quad \ell(x) = \Gamma(x)/\Gamma(1-x)$$

and the $\Upsilon_{\frac{\gamma}{2}}$ function defined as analytic continuation of the following integral defined for $0 < \Re(z) < Q$

$$\ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\Big(\frac{Q}{2}-z\Big)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2}-z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4})\sinh(\frac{t}{\gamma})} \right) \frac{dt}{t}$$

4 point correlation function (GKRV '20)

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}, g_0}$$

4 point correlation function (GKRV '20)

Theorem: assume

$$\alpha_1 + \alpha_2 > Q \quad \text{and} \quad \alpha_3 + \alpha_4 > Q.$$

Then for $|z| < 1$

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}, g_0}$$
$$= \int_{\mathbb{R}_+} C(\alpha_1, \alpha_2, Q - ip) C(\alpha_3, \alpha_4, Q + ip) |z|^{2(\Delta_{Q+ip} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_{(\Delta_{\alpha_i})_i, p}(z)|^2 dp$$

where $\mathcal{F}_{(\Delta_{\alpha_i})_i, p}$ are the holomorphic conformal blocks.

Conformal blocks

The conformal blocks are holomorphic

$$\mathcal{F}_{(\Delta_{\alpha_i})_i,p}(z) = \sum_{n=0}^{\infty} \beta_n z^n$$

where

- $\beta_n = \sum_{|\nu|,|\bar{\nu}|=n} v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+ip}, \nu) Q^{-1}_{\Delta_{Q+ip}}(\nu, \bar{\nu}) v(\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{Q+ip}, \bar{\nu})$
- $\nu = (k_1 \geq k_2 \geq \dots)$ Young diagram of size $|\nu| = \sum_i k_i$
- $v(\Delta, \Delta', \Delta'', \nu) = \prod_j (k_j \Delta' + \Delta'' - \Delta + \sum_{u<j} k_u)$
- $Q^{-1}_{\Delta_{Q+iP}}(\nu, \bar{\nu})$ Shapovalov matrix with central charge $1 + 6Q^2$, and $Q^{-1}_{\Delta_{Q+iP}}(\nu, \bar{\nu})$ its inverse.

Crossing symmetry

Flip $\alpha_1 \leftrightarrow \alpha_3$ using the conformal map $z \mapsto 1 - z$

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle_{\hat{\mathbb{C}}, g_0} = \langle V_{\alpha_3}(0) V_{\alpha_2}(1-z) V_{\alpha_1}(1) V_{\alpha_4}(\infty) \rangle_{\hat{\mathbb{C}}, g_0}$$

Crossing symmetry

Flip $\alpha_1 \leftrightarrow \alpha_3$ using the conformal map $z \mapsto 1 - z$

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle_{\hat{\mathbb{C}},g_0} = \langle V_{\alpha_3}(0) V_{\alpha_2}(1-z) V_{\alpha_1}(1) V_{\alpha_4}(\infty) \rangle_{\hat{\mathbb{C}},g_0}$$

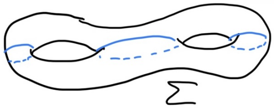
Deduce

$$\int_{\mathbb{R}^+} C(\alpha_1, \alpha_2, Q - ip) C(\alpha_3, \alpha_4, Q + ip) |z|^{2(\Delta_{Q+ip} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_{(\Delta_{\alpha_i})_i, p}(z)|^2 dp$$
$$= \int_{\mathbb{R}^+} C(\alpha_3, \alpha_2, Q - ip) C(\alpha_1, \alpha_4, Q + ip) |1 - z|^{2(\Delta_{Q+ip} - \Delta_{\alpha_3} - \Delta_{\alpha_2})} |\tilde{\mathcal{F}}_{(\Delta_{\alpha_i})_i, p}(1 - z)|^2 dp$$

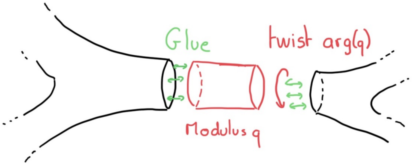
with $\tilde{\mathcal{F}}_{(\Delta_{\alpha_i})_i, p}$ is obtained from $\mathcal{F}_{(\Delta_{\alpha_i})_i, p}$ by flipping $\alpha_1 \leftrightarrow \alpha_3$

Moduli space and plumbing coordinates

- Pants

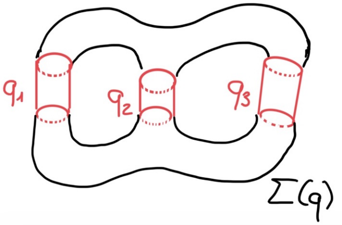


- Gluing annuli



- Local coordinates on $\mathcal{M}_{h,m}$

$$q \in \mathbb{D}^{3h-3+m} \mapsto \Sigma(q)$$



Conformal Bootstrap

Theorem (GKRV '21):

$$\langle V_{\alpha_1}(x_1)\dots V_{\alpha_m}(x_m)\rangle_{\Sigma(q),g} = \int_{\mathbb{R}_+^{3\mathbf{h}-3+m}} \rho(p,\alpha)|\mathcal{F}_{c_L,p,\Delta_\alpha}(q)|^2 \mathrm{d}p$$

- $\rho(p,\alpha)$ product of structure constants
- $\mathcal{F}_{c_L,p,\Delta_\alpha}(q)$ holomorphic series: conformal blocks

Applications or related results

- Conformal bootstrap for Liouville CFT on open surfaces (with Baojun Wu).
  $\hookrightarrow$ random modulus of random planar maps on tori (Ang, Remy, Sun)
- Probabilistic construction of the conformal blocks on tori (Ghosal, Rémy, Sun, Sun)
- Integrability for SLE of CLE (Ang, Holden, Rémy, Sun): Imaginary DOZZ formula, etc...
- Toda CFT (Cerclé, Huang)
- VOA solution for rational CFTs (Y.Z. Huang, ongoing)

Main references

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- Guillarmou, Rhodes, Vargas: Polyakov's formulation of $2d$ bosonic string theory, *Publications Mathématiques IHES* (2019).
- Kupiainen, Rhodes, Vargas: Integrability of Liouville theory: proof of the DOZZ formula, *Annals of Mathematics* (2020).
- Guillarmou, Kupiainen, Rhodes, Vargas: Conformal bootstrap in Liouville Theory, to appear in *Acta Mathematica* (2022).
- Guillarmou, Kupiainen, Rhodes, Vargas: Segal's axioms and bootstrap for Liouville Theory, arXiv:2112.14859.