Landau-Ginzburg models for cominuscule homogeneous spaces

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A pair $(X^\vee, \mathcal{W}_q : X^\vee \to \mathbb{C})$ such that $\mathbb{C}[X^\vee \times \mathbb{C}_q^*]/\langle \partial \mathcal{W}_q \rangle \cong qH^*(X)[q^{-1}]$

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(Assuming the rank of $H_2(X, \mathbb{Z})$ is 1.)

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  - $G^\vee$ is the unique group having the coroots of G as roots and the cocharacters of G as characters

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- Consider $(e_i^\vee)^* : \begin{bmatrix} 1 & b_1 & b_3 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \mapsto b_i$ and $(f_i^\vee)^* : \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_3 & a_2 & 1 \end{bmatrix} \mapsto a_i$
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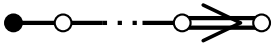
A_n



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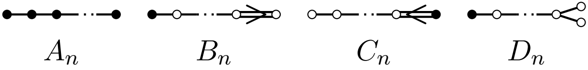

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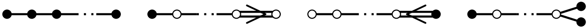

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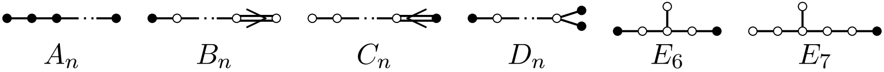

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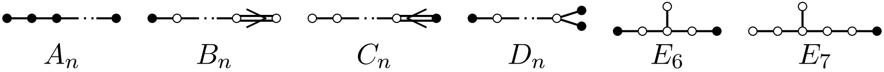

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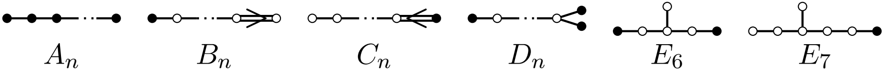

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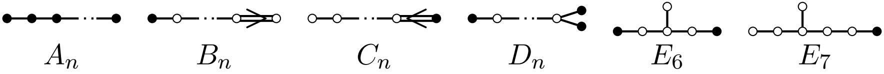

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Summary and more info

- Small quantum cohomology: "counting intersecting curves"
- LG-models $(X^\vee, \mathcal{W}_q)$ satisfying $\mathbb{C}[X^\vee \times \mathbb{C}_q^*]/\langle \partial \mathcal{W}_q \rangle \cong qH^*(X)[q^{-1}]$
- Two LG-models for homogeneous spaces:
  - Lie-theoretic model: any G/P, but abstract
  - Canonical model: correspondence, but only comin. and type-dependent
- More information?
  - More details about the quantum cohomology
  - A third LG-model: Laurent polynomial, type-independent & combinatorial, but only cominuscule & local (+sketch of proof)
  - How to define Plücker coordinates?
  - How to construct the canonical models?
- Related: cluster algebra structures for $\mathbb{C}[X^\vee_{\mathrm{can}}]$

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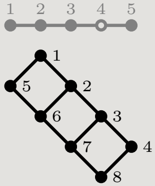

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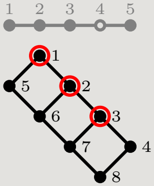

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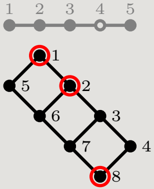

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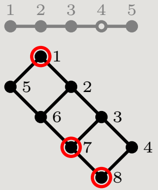

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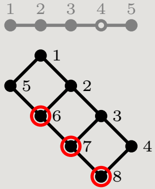
- We can find all these subsets using “moves” on a given one

Example: $\mathrm{Gr}(4,6) = \mathrm{SL}_6/P_4$ (type A_5)

$w^P = (s_2 s_3 s_4 s_5)(s_1 s_2 s_3 s_4)$, $w' = s_2 s_3 s_4$



$r(a_i) = a_1 a_2 a_3 + a_1 a_2 a_8 + a_1 a_7 a_8 + a_6 a_7 a_8$

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- Using that $V(\omega_k^\vee)$ is a *minuscule* representation, we can exactly compute this contribution, as its structure is well-known and expressed type-independently

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$$h^i * h^j = \sum_{d=0}^{\infty} \sum_{k=0}^{n} \langle h^i \cdot h^j \cdot h^{n-k}\rangle_d \, h^k \, q^d$$

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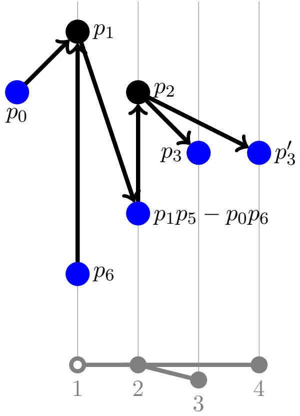

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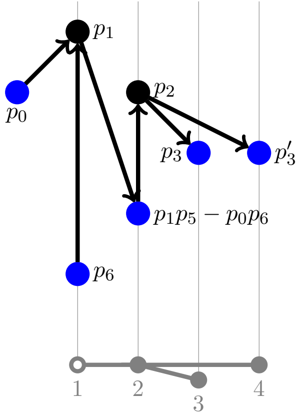

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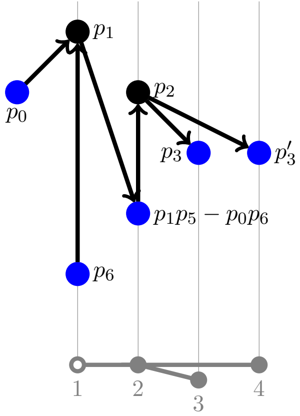

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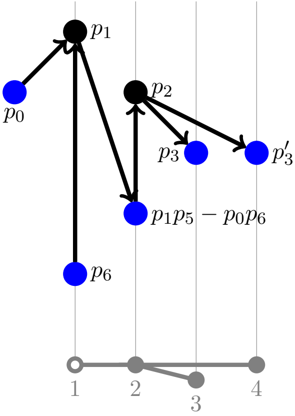

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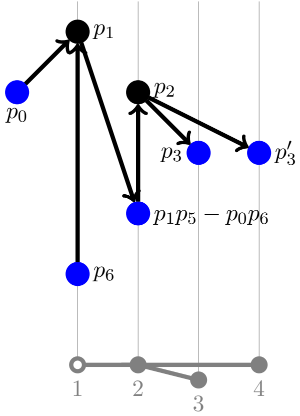

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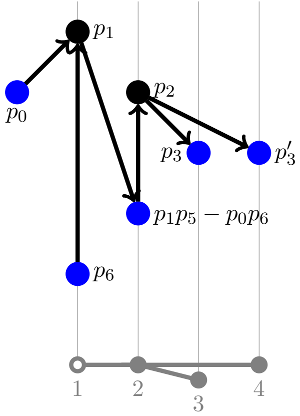

Cluster algebra structure of $\mathbb{C}[X_{\mathrm{can}}^{\vee}]$

- Example: $Q_6 = \mathrm{Spin}_8/P_1$ of type D_4
- $\mathbb{C}[X_{\mathrm{can}}^{\vee}] = \mathbb{C}[p_0^{\pm 1}, p_1, p_2, p_3^{\pm 1}, (p_3')^{\pm 1}, p_4, p_5, p_6^{\pm 1}][(p_1p_5 - p_0p_6)^{-1}]$
- $\mathbb{C}[X_{\mathrm{can}}^{\vee}]$ has a cluster algebra structure:
- $\mu(p_1) = \frac{1}{p_1}\big(p_0p_6 + (p_1p_5 - p_0p_6)\big) = p_5$
- Frozen variables are denominators of $\mathcal{W}_{\mathrm{can}} = \frac{p_1}{p_0} + \frac{p_2p_5}{p_1p_5 - p_0p_6} + \frac{p_4}{p_3} + \frac{p_4}{p_3'} + q\frac{p_1}{p_6}$
- Requires a "Plücker relation" $p_3p_3' - p_2p_4 + p_1p_5 - p_0p_6 = 0$
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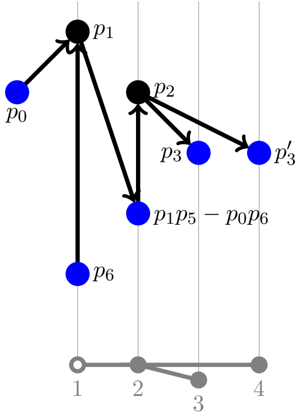
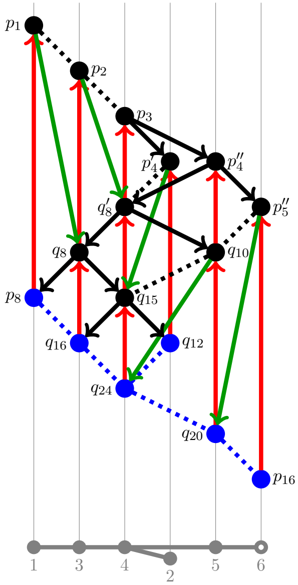
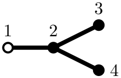
- We constructed cluster structures for the exceptional family



- For E_6/P_6, the mirror has the following cluster structure:



Conjectural construction of cluster structure

- Example: $Q_6 = G/P$ for G of type D_4: 

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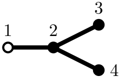
- Example: $Q_6 = G/P$ for G of type D_4: 

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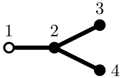
- Example: $Q_6 = G/P$ for G of type D_4: 

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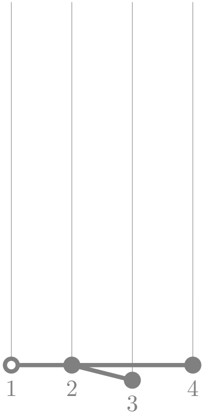

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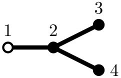
- Example: $Q_6 = G/P$ for G of type D_4: 

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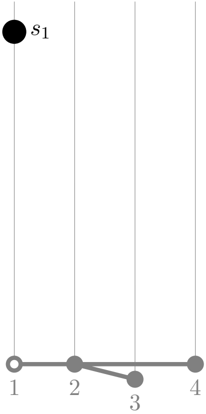

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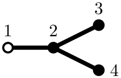
- Example: $Q_6 = G/P$ for G of type D_4: 

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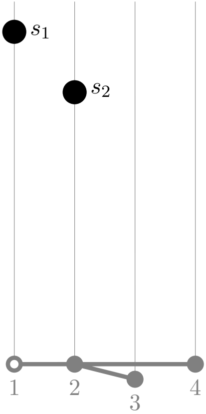

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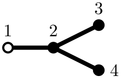
- Example: $Q_6 = G/P$ for G of type D_4: 
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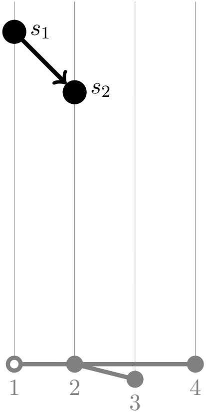

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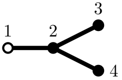
- Example: $Q_6 = G/P$ for G of type D_4: 
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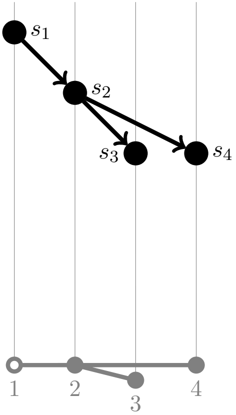

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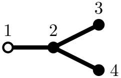
- Example: $Q_6 = G/P$ for G of type D_4: 

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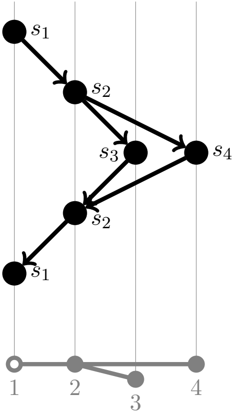

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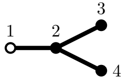

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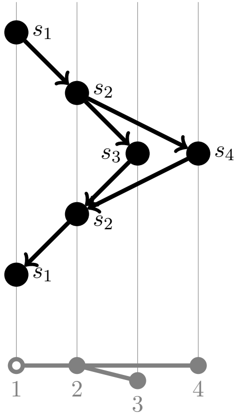

- Quiver due to Perrin (2007)

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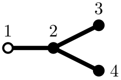
- Example: $Q_6 = G/P$ for G of type D_4: 
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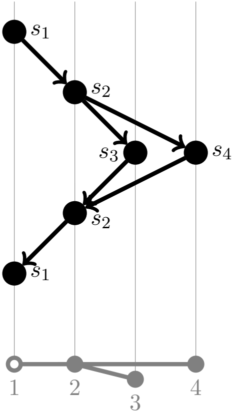

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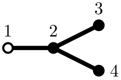
- Example: $Q_6 = G/P$ for G of type D_4: 

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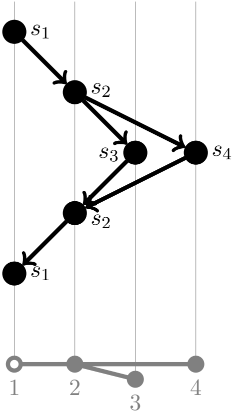

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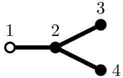
- Example: $Q_6 = G/P$ for G of type D_4: 

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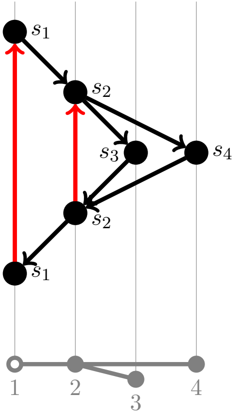

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- Conjectural construction:
  - Draw arrows up in the columns

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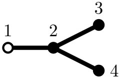

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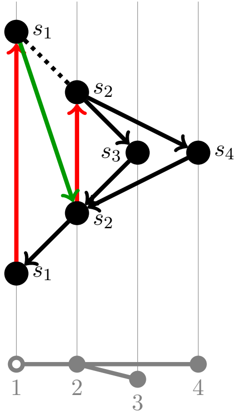

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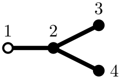
- Example: $Q_6 = G/P$ for G of type D_4: 
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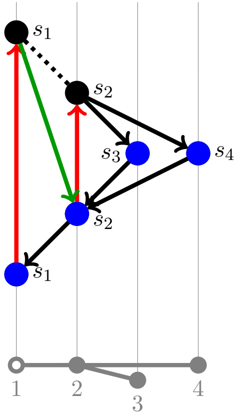

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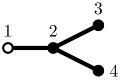

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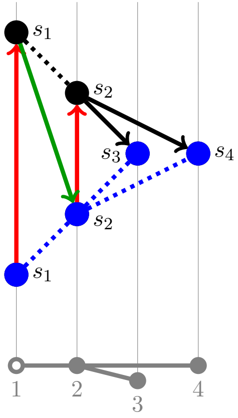

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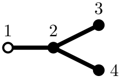
- Example: $Q_6 = G/P$ for G of type D_4: 
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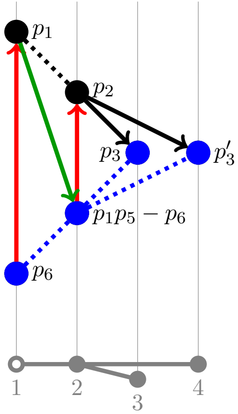

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Summary and more info

- Small quantum cohomology: "counting intersecting curves"
- LG-models $(X^\vee, \mathcal{W}_q)$ satisfying $\mathbb{C}[X^\vee \times \mathbb{C}_q^*]/\langle \partial \mathcal{W}_q \rangle \cong qH^*(X)[q^{-1}]$
- Two LG-models for homogeneous spaces:
  - Lie-theoretic model: any G/P, but abstract
  - Canonical model: correspondence, but only comin. and type-dependent
- More information?
  - More details about the quantum cohomology
  - A third LG-model: Laurent polynomial, type-independent & combinatorial, but only cominuscule & local (+sketch of proof)
  - How to define Plücker coordinates?
  - How to construct the canonical models?
- Related: cluster algebra structures for $\mathbb{C}[X^\vee_{\mathrm{can}}]$