Counting BPS states with discrete charges in M-theory

Thorsten Schimannek

based on 2108.0931 [TS] and work in progress w/ S. Katz, A. Klemm, E. Sharpe

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String Math 2022, Warsaw

Start with smooth
Calabi-Yau 3-fold
$r = h^{1,1}(X_{\text{smooth.}})$

X_{smooth}
$G_{5d} = U(1)^r$

degeneration
(cplx. str.)

$X_{\text{sing.}}$
$G_{5d} = U(1)^r \times \mathbb{Z}_N$

Degeneration develops **nodal singularities** that can not be crepantly resolved *algebraically*.
M-theory develops $\mathbb{Z}_N$ gauge symmetry

Example: Jacobian of smooth genus one fibration

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cplx. Kähler deformation

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Discrete symmetries in M-theory

- M-theory on CY 3-fold X has $U(1)^{b_2(X)}$ gauge symmetry
- Extra $\mathbb{Z}_N$ gauge symmetry arises from **N-torsion** in
  [Camara,Ibanez,Marchesano'11]

$$\mathrm{Tors}\, H_2(X) \simeq \mathrm{Tors}\, H^3(X)$$

- Charged particles from M2-branes wrapping torsion curves

But $\mathbb{Z}_N$ can also arise from singularities!

Recall conifold transition

- M-theory on **conifold singularity** (“node”, ODP)

$$\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\} \subset \mathbb{C}^4$$

  exhibits $U(1)$ gauge symmetry

- **Blow-up** gives resolved conifold $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1$, corresponds to **Coulomb branch** with $U(1)$ preserved

- **Deformation** $x_1^2 + \ldots + x_4^2 = \epsilon$ also smooth, corresponds to **Higgs branch** where $U(1)$ is broken

Local picture in compact CY 3-folds. Always possible globally?

Consider CY 3-fold X with nodal singularities

- Resolution is $\pi : \widehat{X} \to X$ is called **crepant** if

$$K_{\hat{X}} = \pi^* K_X$$

- Existence of crepant (i.e. CY) global Kähler resolution depends on intricate global properties **[Clemens'83]**

- If no crepant kähler resolution (CKR) of X exists, **non-Kähler resolutions $\widehat{X}$ will exhibit** **[Werner'87]**

$$\mathrm{Tors}\, H_2(\widehat{X}) \neq 0\,!$$

Conjecture: *(based on F-theory)*
M-theory on X has discrete gauge symmetry Tors $H_2(\widehat{X})$, where $\widehat{X}$ is in general *not Kähler*!

Examples

1. Jacobian fibration $J(X)$ of smooth genus one fibered CY 3-fold X without a section. e.g. **[Braun,Morrison'14]**...
   (beautiful interplay with arithmetic geometry and modularity!) **[TS'21]**

2. **Simplest example (?): [Katz,Klemm,T.S.,Sharpe'xx]**
   Degeneration of generic CY double cover of $\mathbb{P}^3$

$$X = \{w^2 = \det A_{8\times 8}(x_1, \ldots, x_4)\} \subset \mathbb{P}(1,1,1,1,4)$$

Simplest (?) example: **[Katz,Klemm,T.S.,Sharpe'xx]**

1. Start with generic smooth CY double cover of $\mathbb{P}^3$

$$X_{\text{def.}} = \{w^2 = f_8(x_1, \ldots, x_4)\} \subset \mathbb{P}(1,1,1,1,4)$$

$$h^{1,1}(X_{\text{def.}}) = 1\,, \quad h^{2,1}(X_{\text{def.}}) = 149\,, \quad H_2(X_{\text{def.}}) = \mathbb{Z}$$

2. Degenerate to

$$X = \{w^2 = \det A_{8\times 8}(x_1, \ldots, x_4)\} \subset \mathbb{P}(1,1,1,1,4)$$

Now X has 84 nodes and **no crepant Kähler resolution**! **[Addington'09]**

3. Can construct **non-Kähler crepant resolution** $\hat{X}$ with $H_2(\hat{X}) = \mathbb{Z} \oplus \mathbb{Z}_2$. **[Addington'09]**, **[Katz,Klemm,T.S.,Sharpe'xx]**

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Multiplicities of $\mathbb{Z}_N$ charged BPS states?

Five-dimensional **BPS particles arise from M2 branes wrapping curves** in the Calabi-Yau. The multiplicities are encoded in the (A-model) **topological string partition function** as **Gopakumar-Vafa invariants** n_β^g:

$$\log\, Z^X_{\text{top.}}(\omega) = \sum_{\beta\in H_2(X)} \sum_{g=0}^{\infty} \sum_{m=1}^{\infty} c(g,m,\lambda)\cdot n_\beta^g \exp\left(2\pi i m \int_\beta \omega\right)$$

- $\omega = B + iV$ cplx. Kähler form
- λ topological string coupling
- $\beta \in H_2(X) \simeq U(1)^r \times \mathbb{Z}_N$ charge
- $c(g,m,\lambda) = \frac{1}{m}\left[2\sin\left(\frac{m\lambda}{2}\right)\right]^{2g-2}$

How can one resolve discrete charges?

Problem: Integral $\int_\beta \omega$ vanishes for $\beta \in \text{Tors}\, H_2$.

Solution: replace $\exp\left(2\pi i\int_\beta \omega\right)$ by homomorphism **[Aspinwall,Morrison,Gross'95]**

$$e^{-S}:\ H_2(X,\mathbb{Z}) = \mathbb{Z}^r \oplus \mathbb{Z}_N \to \mathbb{C}^*\,.$$

Can use $\omega \in H^2(X,\mathbb{C})$ and $k \in \{0,\ldots,N-1\}$ to write

$$b_{\omega,k}(\beta,1) = e^{\frac{2\pi i k}{N}} \exp\left(2\pi i \int_\beta \omega\right)\,.$$

Turning on $\frac{k}{N}$ B-field should give partition function

$$\log\, Z^X_{\text{top.}}(\omega,k) = \sum_{\beta\in\mathbb{Z}^r}\sum_{q_{\mathbb{Z}_N}=0}^{N-1}\sum_{g=0}^{\infty}\sum_{m=1}^{\infty} c(g,m,\lambda)\cdot n^g_{\beta,q_{\mathbb{Z}_N}} b_{\omega,k}(\beta,q_{\mathbb{Z}_N})$$

(see also **[Braun,Kreuzer,Ovrut,Scheidegger'07]**, **[Dedushenko,Witten'14]** for smooth case)

But $\hat{X}$ is not Kähler. Are we in trouble?

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Let's go back to conifold!

Non-commutative resolutions

Conifold has **non-commutative crepant resolution** (NCCR)

[Van der Bergh'04], **[Szendrői'08]**

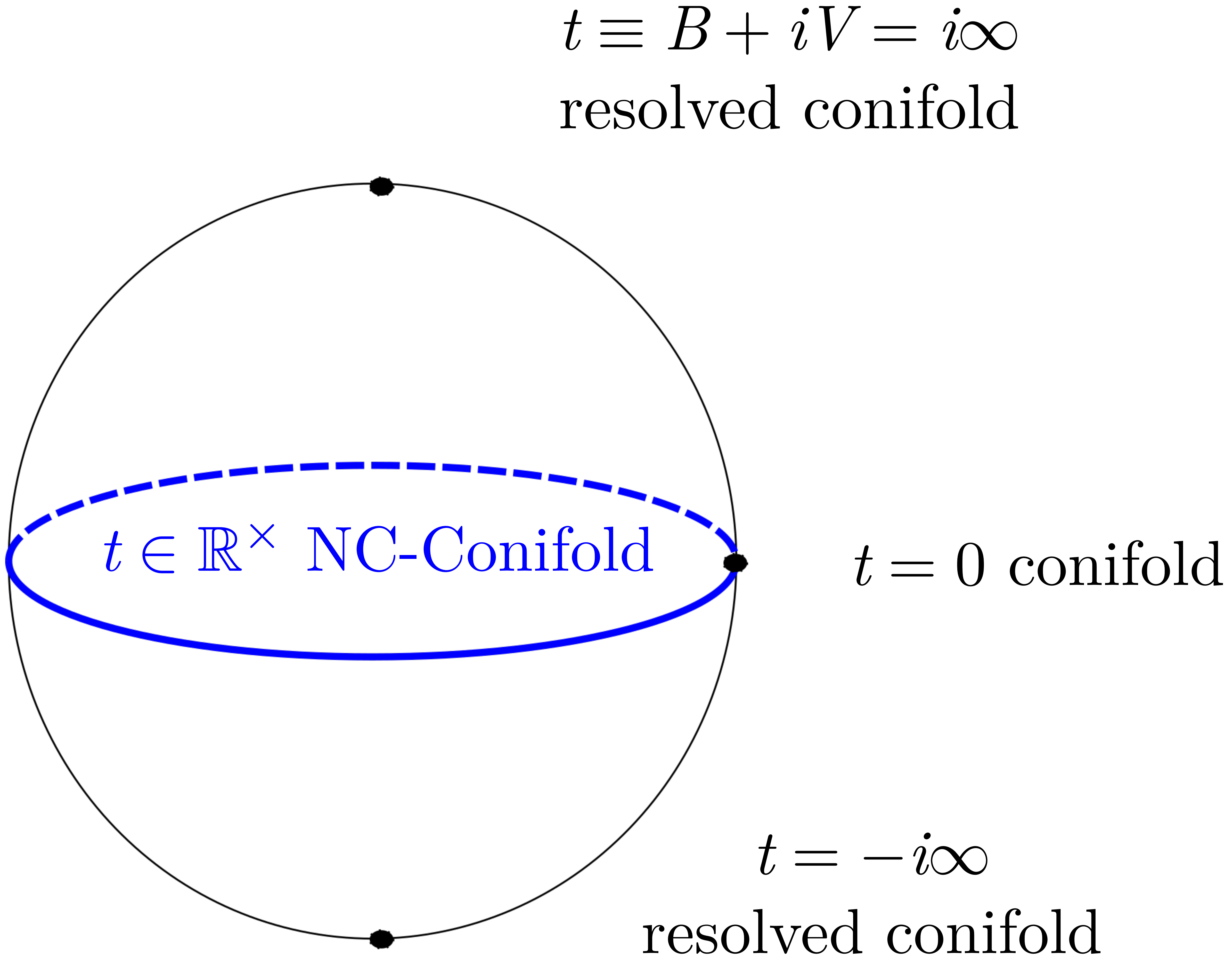
Usual Algebraic Geometry: Correspondence between geometry and commutative ring of functions

$$\{x_1^2 + \ldots + x_4^2 = 0\} \subset \mathbb{C}^4 \quad \leftrightarrow \quad R = \mathbb{C}[x_1, \ldots, x_4]/\langle x_1^2 + \ldots + x_4^2 \rangle$$

NCCR: Non-commutative algebra that contains R as center. Can define notion of crepancy and smoothness.

NC-Conifold given by Jacobi algebra of quiver with potential!

NC-Conifold is realized at equator of str. Kähler moduli space
[Szendrői'08]



Can define NC-Donaldson-Thomas/Gopakumar-Vafa invariants!

It turns out that this **picture makes sense globally**!
[T.S.'21], **[Katz,Klemm,T.S.,Sharpe'XX]**

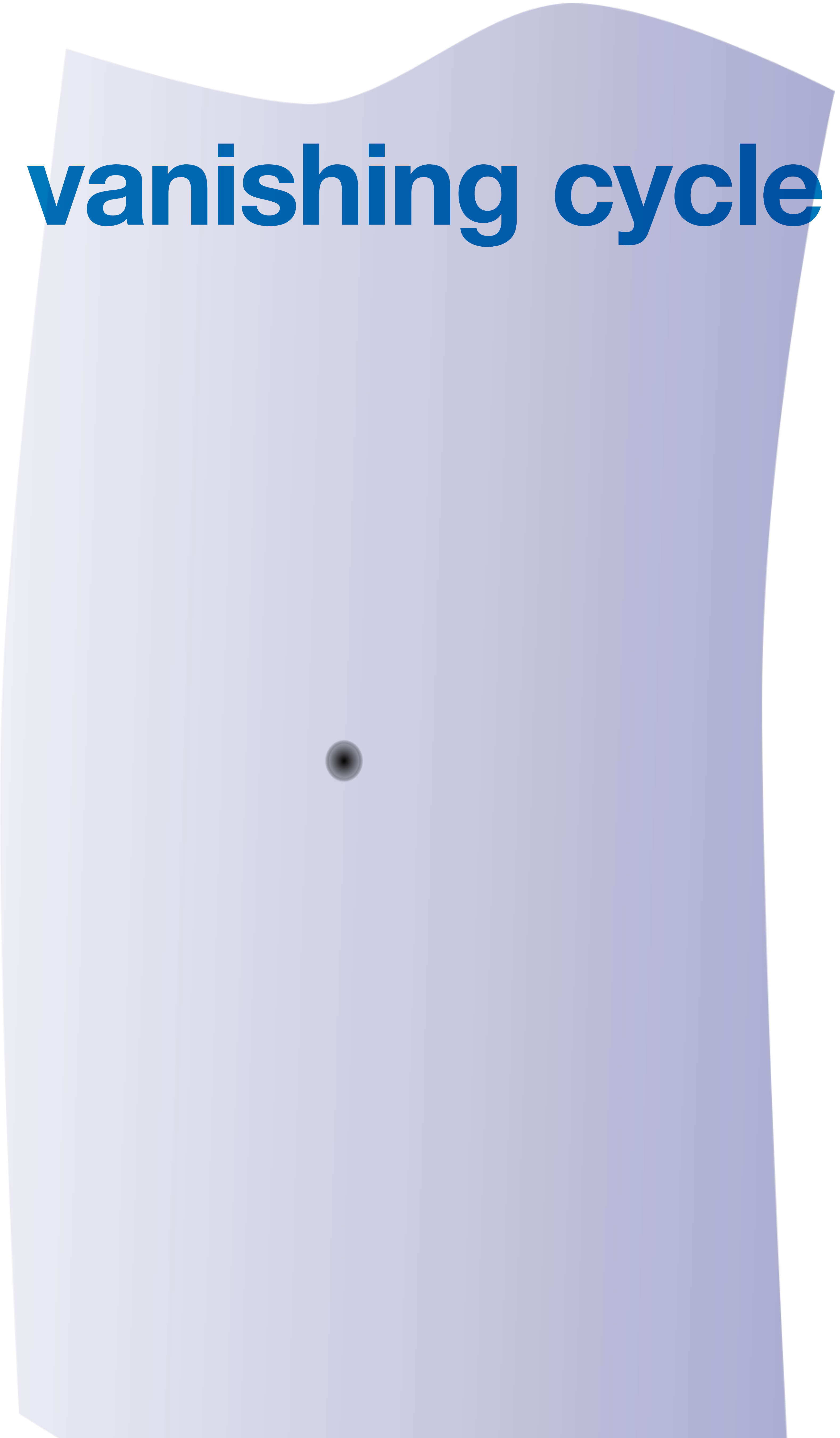
Recall $X = \{w^2 = \det A_{8\times 8}(x_1, \ldots, x_4)\} \subset \mathbb{P}(1,1,1,1,4)$

- Kuznetsov's non-commutative resolution $X_{\text{n.c.}}$ of X at hybrid point in GLSM of CICY $X_{2,2,2,2} \subset \mathbb{P}^7$ **[Caldararu,Distler,Hellerman,Pantev,Sharpe'07]**
  Homologically projective dual to $X_{2,2,2,2}$ **[Kuznetsov'06]**
- Hori/Seiberg dual to $U(1) \times SO(8)$ GLSM + theta angle **[Hori'11]**
  Locally around nodes reduces to conifold with $\frac{1}{2}$ *B-field!*
- Can study top. string on $X_{\text{n.c.}}$ using B-model analytic continuation

Larger class of examples from "Clifford double mirrors" **[Borisov,Li'16]**
(Other large source of nc-resolutions from torus fibrations **[Caldararu'00]**, **[T.S.'21]***)*

Intuition:
Fractional B-field along vanishing cycle

1. Start with singularity
2. Perform small resolution
   In general non-Kähler!
3. Turn on **B-field**
4. Shrink 2-cycle back to point



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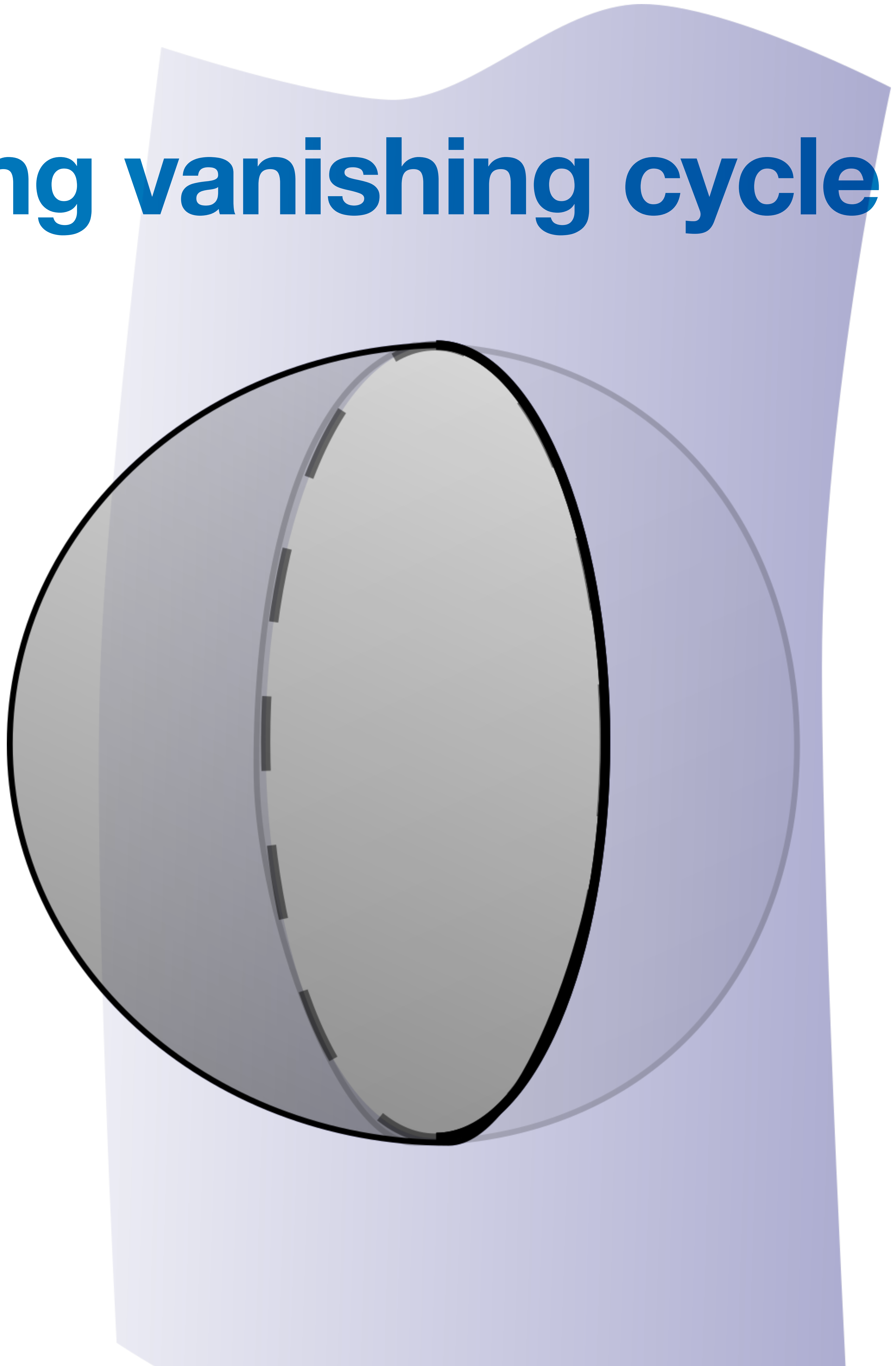

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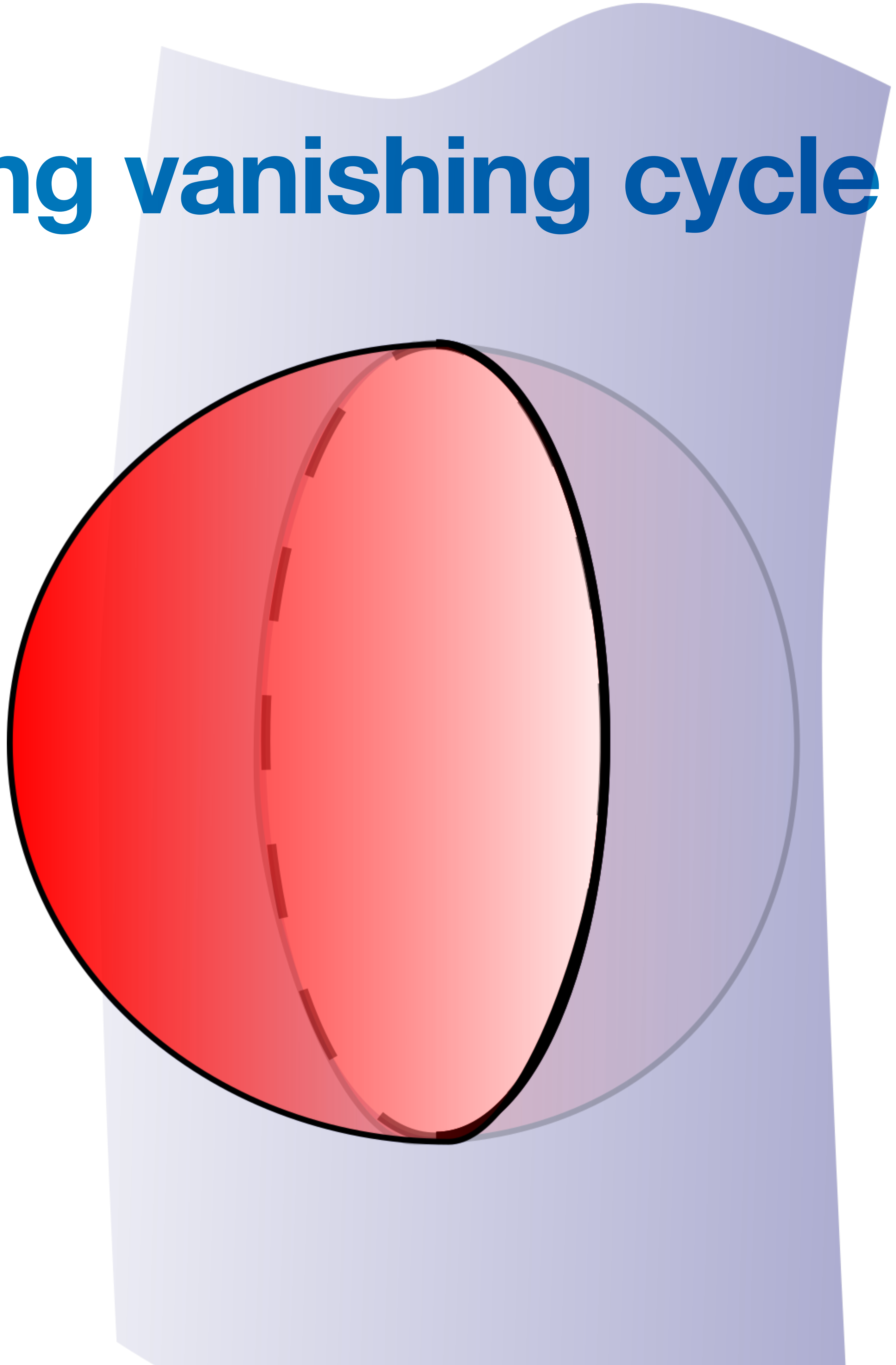

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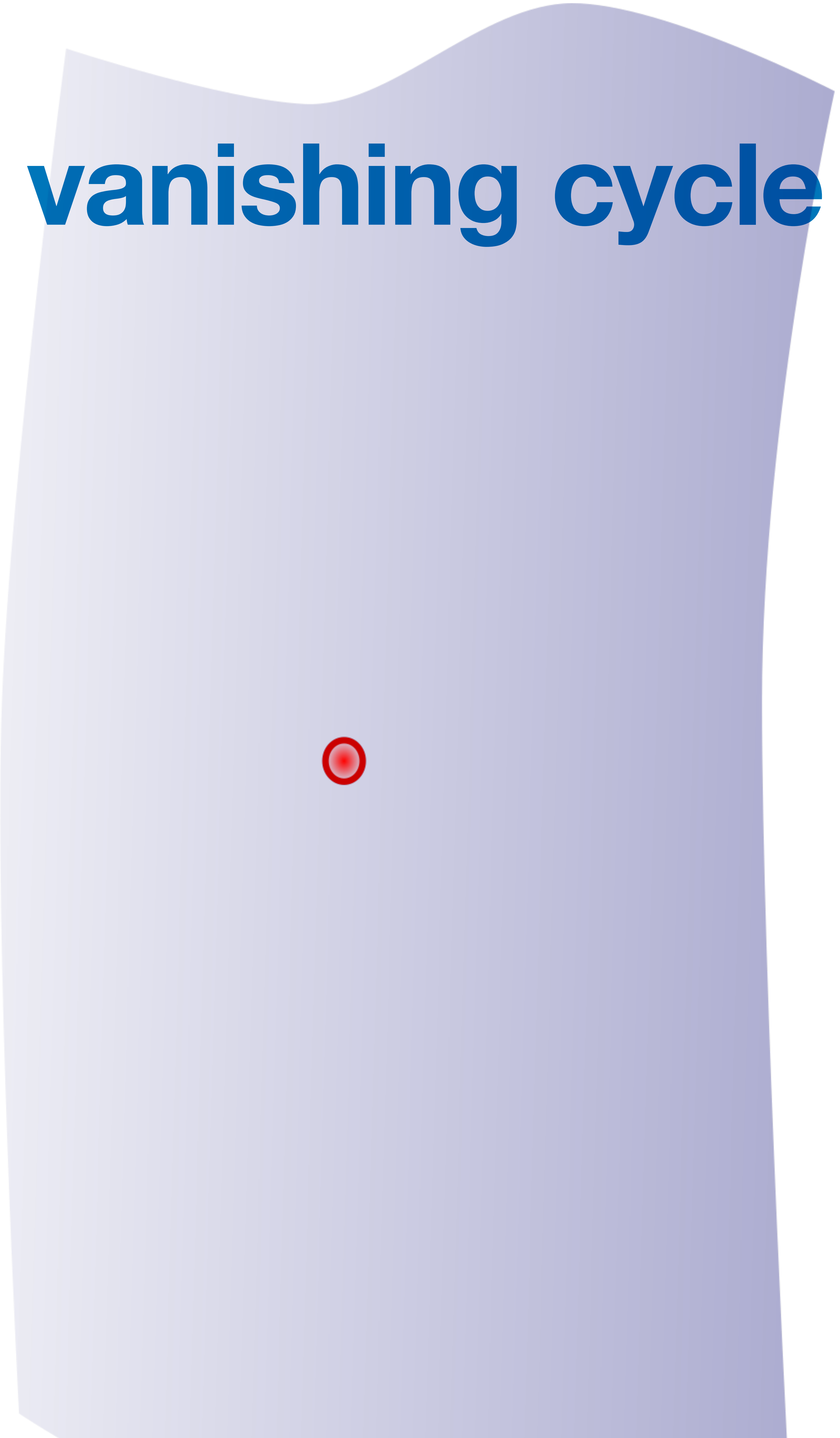

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Topological string on $X_{n.c.}$ exists and
allows us to calculate $\mathbb{Z}_N$ refined Gopakumar-Vafa invariants!

$U(1)$ GV-invariants for $X = \{w^2 = f_8(x_1, \ldots, x_4)\} \subset \mathbb{P}(1,1,1,1,4)$

n_g^β	$\beta = 1$	2	3	4
$g = 0$	29504	128834912	1423720546880	23193056024793312
1	0	41312	21464350592	1805292092705856
2	0	864	−16551744	12499667277744
3	0	6	−177024	−174859503824
4	0	0	0	396215800
5	0	0	0	301450
6	0	0	0	4152
7	0	0	0	24

$U(1) \times \mathbb{Z}_2$ GV-invariants for $X = \{w^2 = \det A_{8\times 8}(x_1, \ldots, x_4)\} \subset \mathbb{P}(1,1,1,1,4)$

Extracted from $Z_{top.}(X_{def.})$ + $Z_{top.}(X_{n.c.})$!

$n_g^{\beta,0}$	$\beta = 1$	2	3	4
$g=0$	14752	64415616	711860273440	1159652800434432 0
1	0	20160	10732175296	902646044328864
2	0	504	−8275872	6249833130944
3	0	0	−88512	−87429839184
4	0	0	0	198065872
5	0	0	0	157306
6	0	0	0	1632
7	0	0	0	24

$n_g^{\beta,1}$	$\beta = 1$	2	3	4
$g=0$	14752	64419296	711860273440	1159652802044899 2
1	0	21152	10732175296	902646048376992
2	0	360	−8275872	6249834146800
3	0	6	−88512	−87429664640
4	0	0	0	198149928
5	0	0	0	144144
6	0	0	0	2520
7	0	0	0	0

Can be interpreted as enumerative invariants of non-Kähler $\hat{X}$!

[Katz,Klemm,T.S.,Sharpe'XX]

Summary

- Can calculate $\mathbb{Z}_N$ refined GV-invariants from $Z_{\text{top.}}$ on singular Calabi-Yau 3-folds and their non-commutative crepant resolutions
  [**T.S.'21**], [**Katz,Klemm,T.S.,Sharpe'XX**]

- For $\mathbb{Z}_5$ this gives integral invariants from irrational partition functions! [**T.S.'21**]

- Beautiful modular structure in case of torus fibrations [**T.S.'21**]
  $Z_{\text{top.}}$ *transform as vector valued Jacobi forms under Atkin-Lehner involutions*
  (see also [**Knapp,Scheidegger,T.S.'21**]*)*

- New formula for constant map contributions to $Z_{\text{top.}}$ on nc-resolutions
  More boundary conditions to solve top. string!
  [**Katz,Klemm,T.S.,Sharpe'XX**]

Thank you for your attention!