Approximations for Calabi-Yau and other special holonomy/structure metrics

FABIAN RUEHLE

StringMath 2022

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Based on:

[Larfors, Lukas, FR, Schneider: 2111.01436, 2205.13408]

[Anderson, Gray, Gerdes, Krippendorf, Raghuram, FR: 2012.04656]

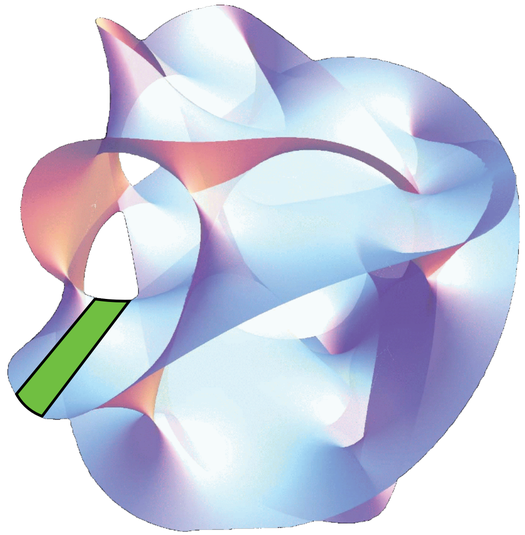
[Ashmore, FR: 2103.07472]

[FR: Phys.Rept. 839 (2020)]

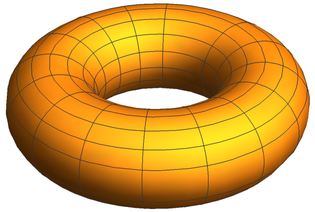
Outline

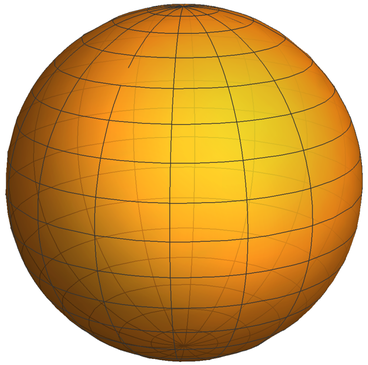
- History and Motivation
- Approximating CY metrics
  - Finding points on Calabi-Yau manifolds
  - Approximating Calabi-Yau metrics
- Examples:
  - One-parameter family of Fermat Cubic
  - Massive string spectrum for the Quintic CY threefold
- Conclusion

$$y_{ijk}\Phi^i\Phi^j\Phi^k$$



$$g_{i\bar{j}}F^{i\bar{j}} = 0$$





$$\mu = \frac{1}{\mathrm{rk}(V)}\int_X J\wedge J\wedge \mathcal{F}$$

$$\Box_{6D}\varphi_{6D} = \lambda\varphi_{6D} = \sqrt{\det(g)}^{\,-1}\partial_a\sqrt{\det(g)}g^{ab}\partial_b$$

Motivation and previous work

CY metrics and the Monge-Ampere equation

- By the **Calabi-Yau theorem**, for compact Kähler manifolds with $c_1(X) = 0$, there exists a **unique Ricci flat metric** in the same Kähler class as a chosen reference metric g

[Calabi `54, `57; Yau `77, `78]

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- Note: Instead of approximating Φ (CY Kähler potential correction), we **could also approximate** g_{CY} **directly** (subject to its Kähler form being closed,…)

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  - H is a hermitian matrix that satisfies the **condition** $(H^{-1})^{I\bar{J}} = \frac{1}{\mathrm{vol}(\mathrm{X})} \int_X \frac{s^I \bar{s}^{\bar{J}}}{s^\dagger \cdot H \cdot s}$
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- In the limit $k \to \infty$, the **balanced metric** becomes the **Ricci-flat metric**

[Donaldson `01; Donaldson 05]

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  [Headrick, Nassar `09; Ashmore, He, Ovrut `19; Cui, Gray `19; Anderson, Gray, Gerdes, Krippendorf, Raghuram, FR `20; Douglas, Lakshminarasimhan, Qi `20; Jejjala, Mayorga Pena, Mishra `20; Larfors, Lukas, FR, Schneider `21 `22]

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- **Kachru et.al.** construct **Ricci-flat metrics on K3s** (given as toroidal orbifolds), but these methods do not generalize in an obvious way to $d > 2$
[Kachru, Tripathy, and Zimet `18, `20; Tripathy, Zimet `20]

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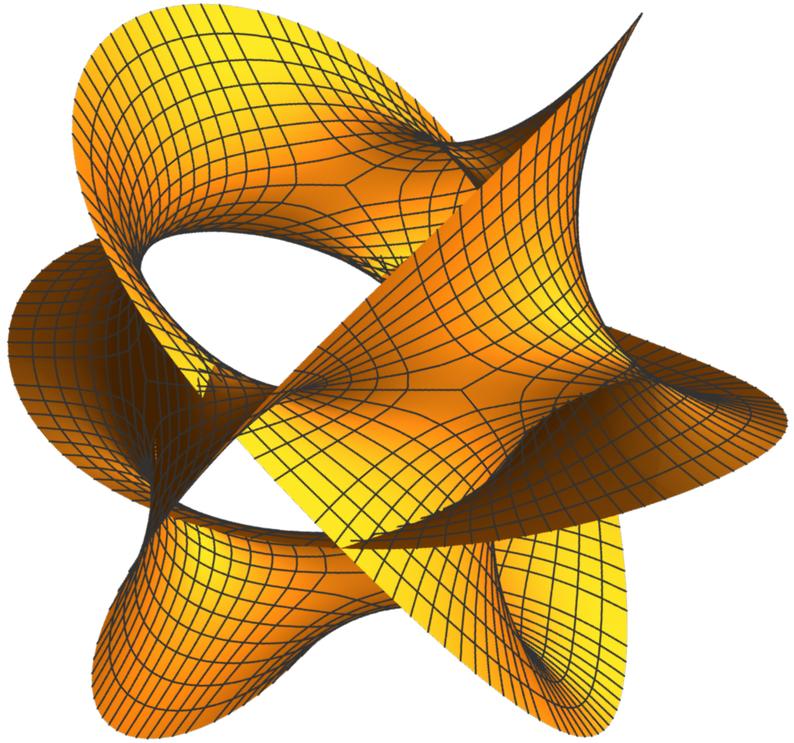
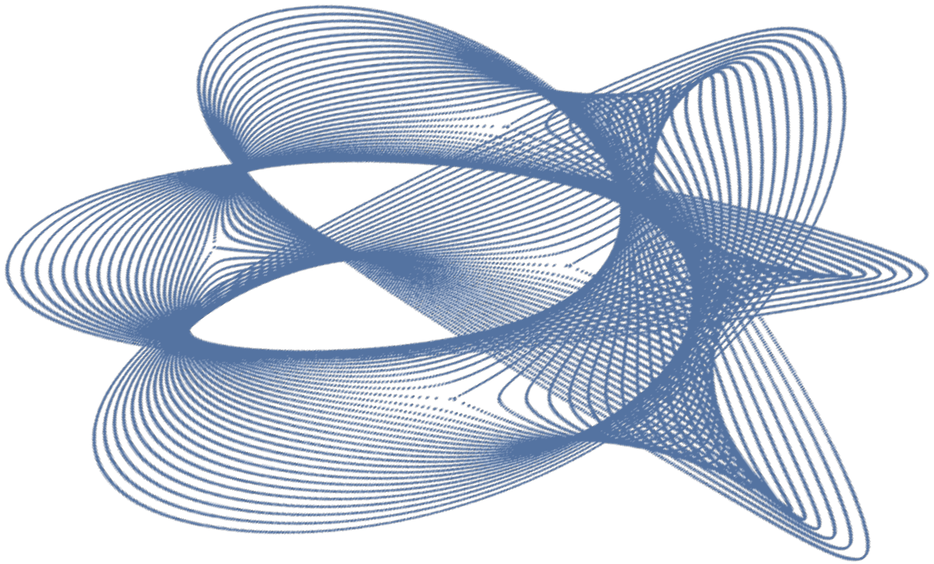
- Can we learn something about
  - **Mirror symmetry** / **SYZ** conjecture? [Strominger, Yau, Zaslow `96]
  - **Existence** of **Ricci-flat** metrics for **SU(n) / G2 / SO(n) holonomy / structure** manifolds, for which no existence proof is known?





Points on CY varieties

Finding points on the CY

- Often, string theorists like to explore **two types of CY constructions**:
  - **Complete Intersections** in products of projective ambient spaces [Candelas, Dale, Lutken, Schimmrigk `88; Green, Hubsch, Lutken `89]
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- This is too naive, since we will **not know how the points will be distributed** after restriction to the Calabi-Yau

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- One can endow $\mathbb{P}H^0(J^{(\alpha)}, \mathcal{A})$ with an **FS metric** obtained from the Kähler potential $K^{(\alpha)} = \log \sum_{i,j=0}^{r_\alpha} \bar{s}_{\bar{i}}^{(\alpha)} H^{\bar{i}j} s_j^{(\alpha)}$, H hermitian (we take $H = \mathbb{1}$)

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- A **different point sampling** method for toric varieties based on a tropical sampling method for evaluating Feynman integrals was recently discussed in Borinski et.al.

[Borinsky, Sattelberger, Sturmfels, Telen `22]

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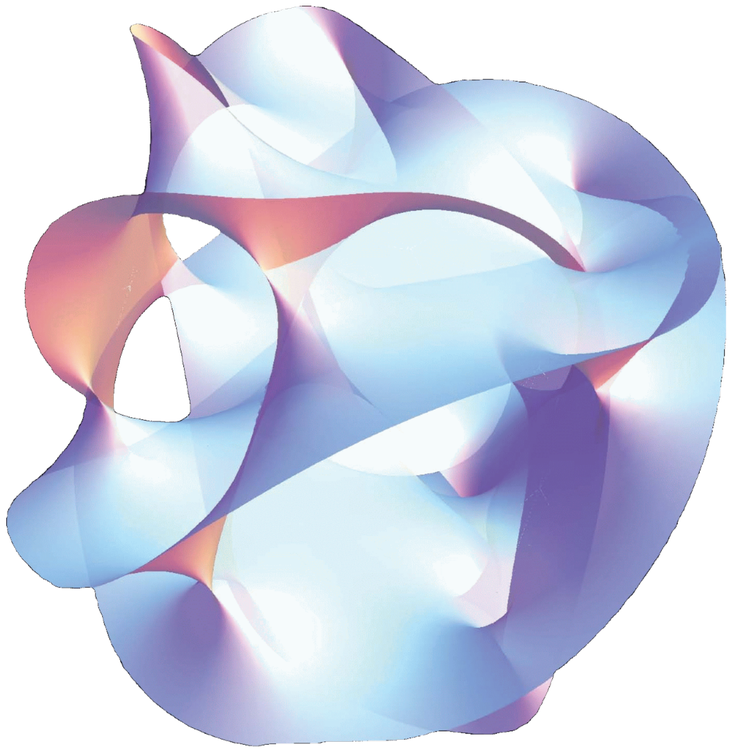
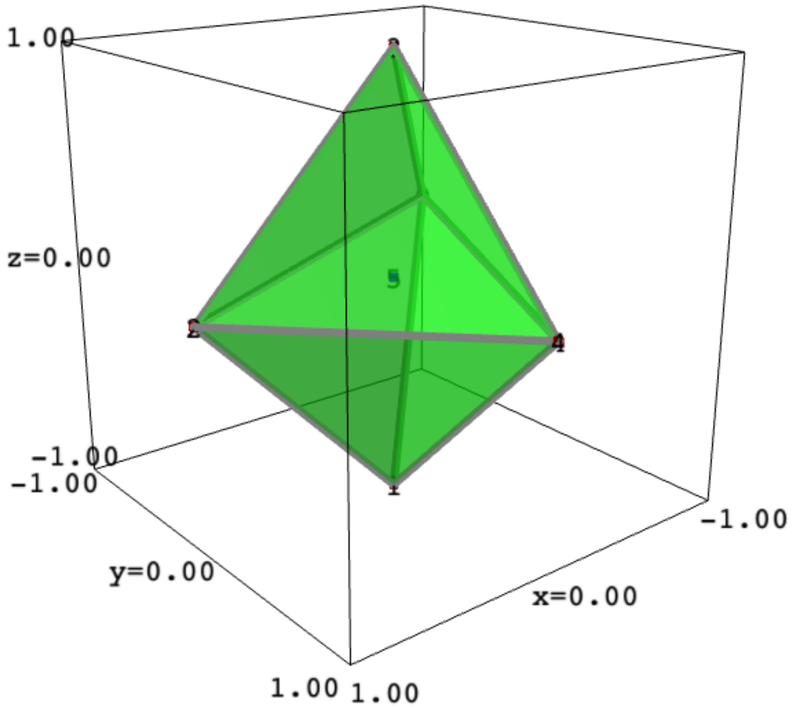
- **But:**
  - Dimensionality of ambient space increased: $\sum_i r_i > D$
  - Sections not independent

$\Rightarrow$ **CY** is a **non-complete intersection** in a **product of projective spaces**





CY metrics for toric varieties and CICYs

Approximating the metric

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- By a theorem of **Cybenko** in `89 (which has been further generalized since), **NNs** are, under mild assumptions, **universal function approximators** [Cybenko `89; …; Kidger, Lyons `19]

Neural Networks

- NNs **parameterize maps** $f : \mathbb{R}^{n_{\text{in}}} \to \mathbb{R}^{n_{\text{out}}}$ as compositions of **affine maps** $f_i : \mathbb{R}^{n_{i-1}} \to \mathbb{R}^{n_i}$ and **component-wise non-linear functions** $\sigma_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$

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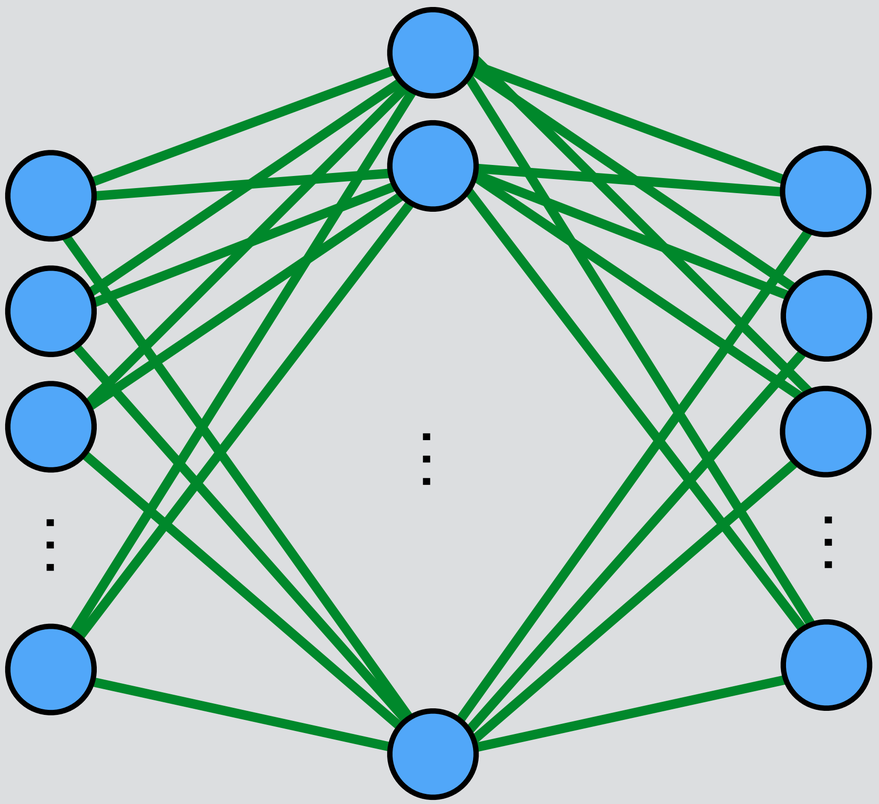

$$z_\alpha^{\text{out}} = \sigma_{\text{out}}(W_\alpha^{\beta,\text{out}} \sigma_1(W_\beta^{\gamma,\text{in}} z_\gamma^{\text{in}} + b_\beta^{\text{in}}) + b_\alpha^{\text{out}})$$

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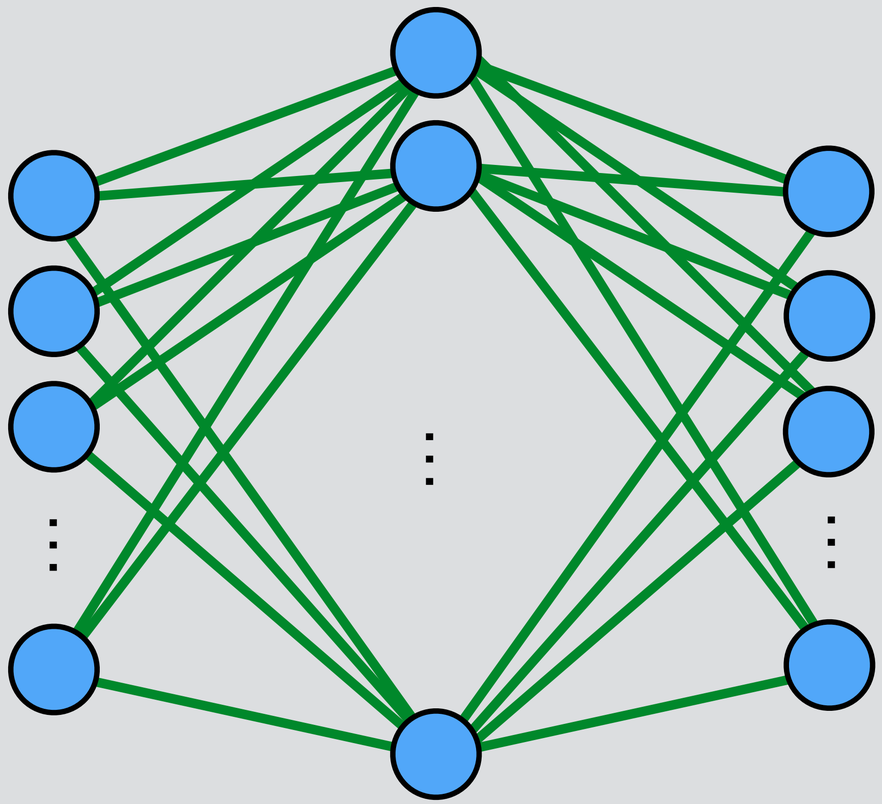

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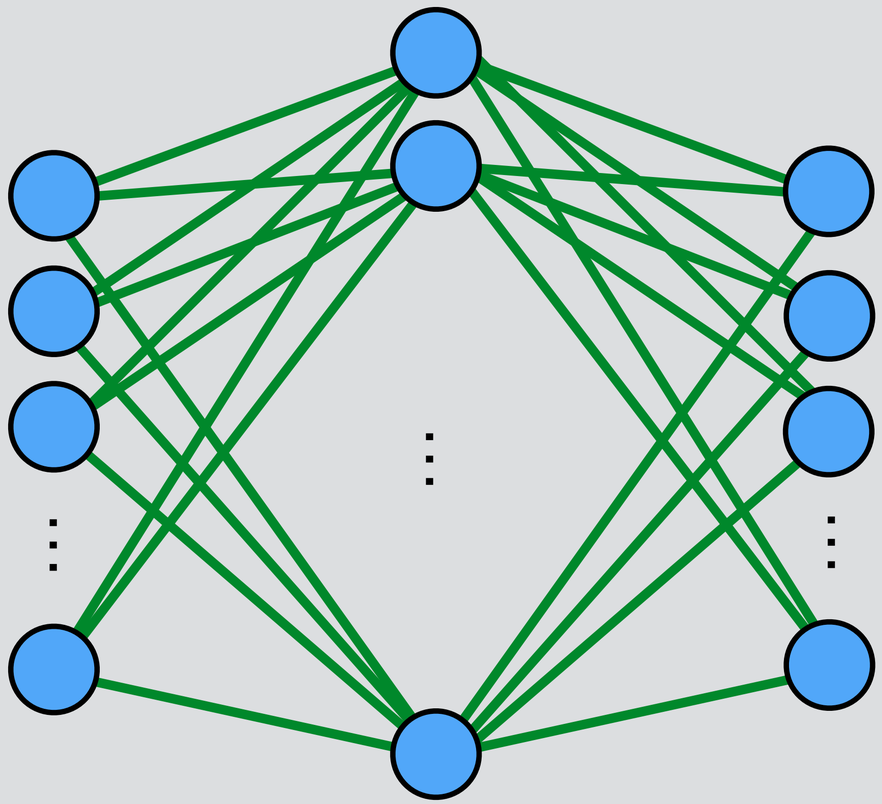
- For not fully understood reasons, this combination of parameterizing a map **works unreasonably well**. This is at the forefront of all ML breakthroughs in recent years and an active topic of research in CS



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Metric approximations

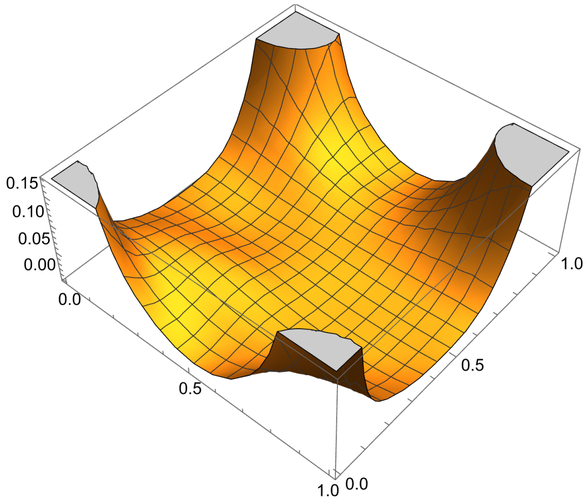
- Several conceivable ways to **approximate the metric**

Name	Ansatz
Free	$g_{\mathrm{pr}} = g_{\mathrm{NN}}$
Additive	$g_{\mathrm{pr}} = g_{\mathrm{FS}} + g_{\mathrm{NN}}$
Multiplicative, element-wise	$g_{\mathrm{pr}} = g_{\mathrm{FS}} + g_{\mathrm{FS}} \odot g_{\mathrm{NN}}$
Multiplicative, matrix	$g_{\mathrm{pr}} = g_{\mathrm{FS}} + g_{\mathrm{FS}} \cdot g_{\mathrm{NN}}$
ϕ-model	$g_{\mathrm{pr}} = g_{\mathrm{FS}} + \partial\bar{\partial}\phi$

- For **some approximations**, we need to **impose conditions on top of** the **MA equation**, e.g. that the metric is Kahler, that the Kahler class is not changed, and/or that it is well-defined on patch transitions

Different Losses

- Total loss $\mathcal{L} = \alpha_1 \mathcal{L}_{\mathrm{MA}} + \alpha_2 \mathcal{L}_{\mathrm{dJ}} + \alpha_3 \mathcal{L}_{\mathrm{transition}} + \alpha_4 \mathcal{L}_{\mathrm{Ricci}} + \alpha_5 \mathcal{L}_{\mathrm{Kclass}}$
- Monge-Ampere loss $\mathcal{L}_{\mathrm{MA}} = \left|\left| 1 - \frac{1}{\kappa} \frac{\det g_{\mathrm{pr}}}{\Omega \wedge \bar{\Omega}} \right|\right|_n$
- Kähler loss $\mathcal{L}_{\mathrm{dJ}} = \sum_{ijk} ||\mathrm{Re}\, c_{ijk}||_n + ||\mathrm{Im}\, c_{ijk}||_n \ , \qquad c_{ijk} = \partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}}$
- Transition loss $\mathcal{L}_{\mathrm{transition}} = \frac{1}{d} \sum_{\mathcal{U},\mathcal{V}} \left|\left| g_{\mathrm{pr}}^{\mathcal{V}} - T_{\mathcal{U}\mathcal{V}} \cdot g_{\mathrm{pr}}^{\mathcal{U}} \cdot (T_{\mathcal{U}\mathcal{V}})^\dagger \right|\right|_n$
- Ricci loss (redundant since equivalent to MA loss) $\mathcal{L}_{\mathrm{Ricci}} = ||R||_n = \left|\left| \partial\bar{\partial} \ln \det g_{\mathrm{pr}} \right|\right|_n$
- Kähler class loss $\mathcal{L}_{\mathrm{Kclass}} = \frac{1}{h^{1,1}(X)} \sum_{\alpha=1}^{h^{1,1}(X)} \left|\left| \mu_t(\mathcal{O}_X(e_\alpha)) - \int_X J_{\mathrm{pr}}^2 \wedge F_{\mathrm{FS},\alpha} \right|\right|_n$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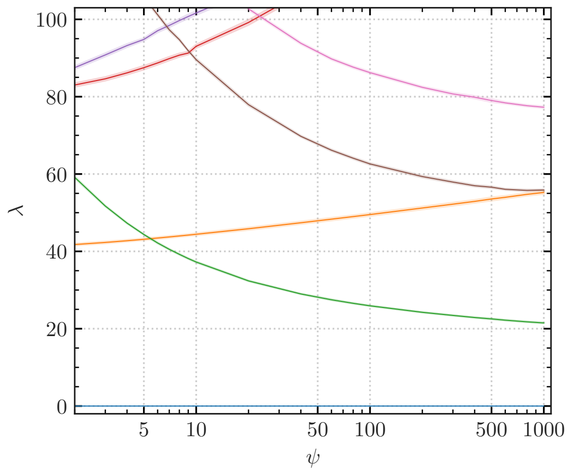
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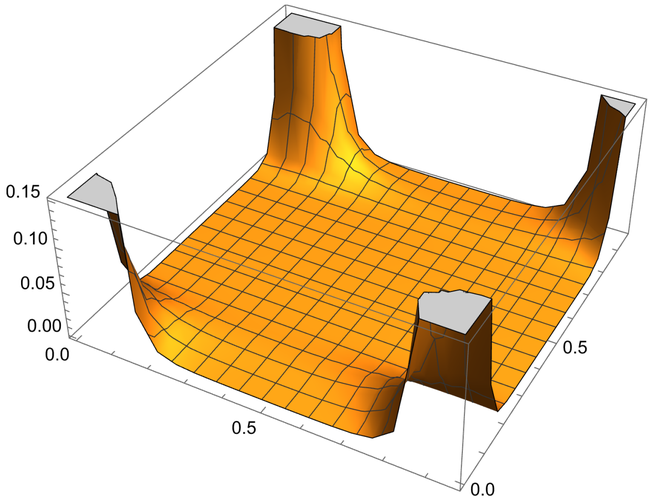
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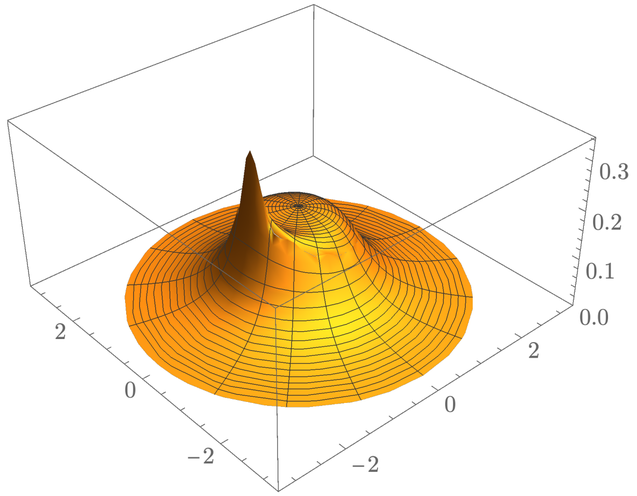
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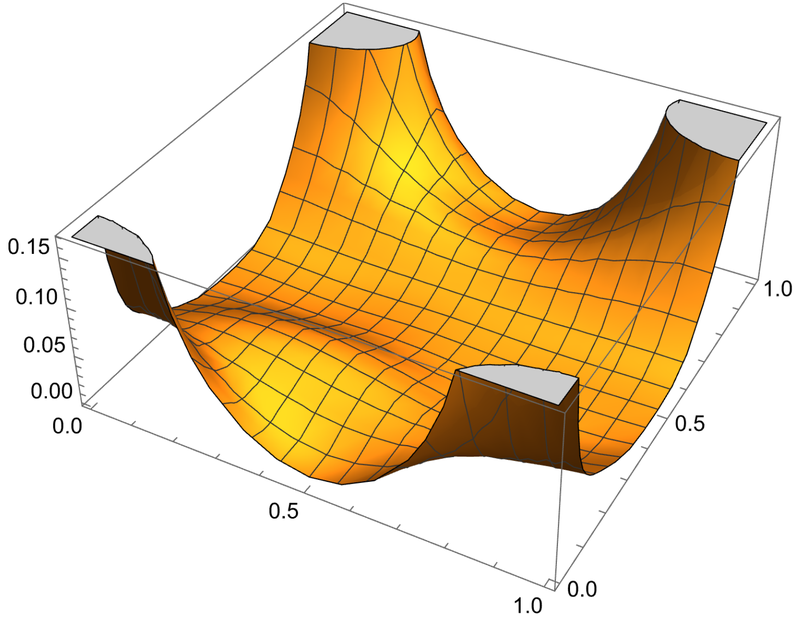
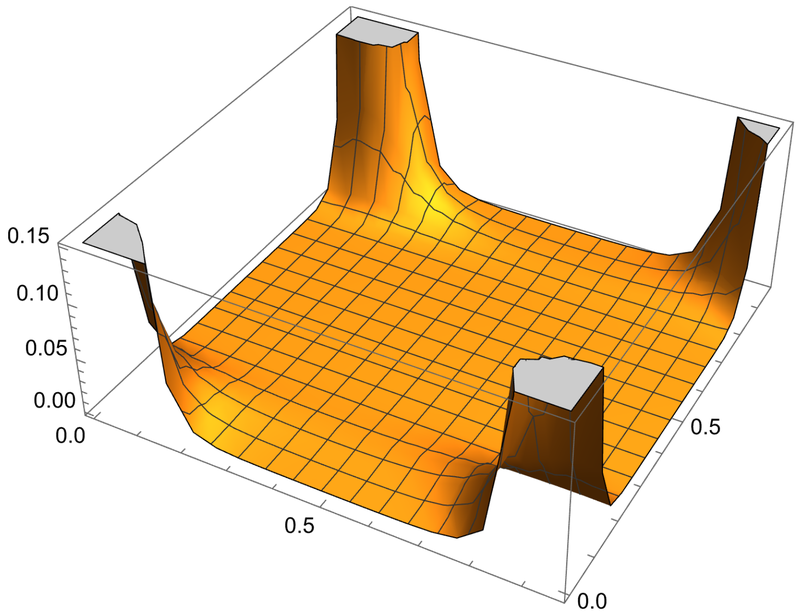
Examples

Example 1: Cubic in $\mathbb{CP}^2$



Pullback of Fubini-Study metric expressed in flat coordinates using Weierstrass $\wp$ function



Approximate CY metric, trained on 10k points with 3 layer NN (around 20 seconds)

$$z_0^3 + z_1^3 + z_2^3 - 3\psi z_0 z_1 z_2 \;\leftrightarrow\; y^2 z = 4x^3 + g_2(\psi) x z^2 + g_3(\psi) z^3 \quad , \quad \tau \approx 1.62i$$

Example 2: Massive KK modes for Quintic

1. **Compute** the **CY metric** at **different** points in **moduli** space

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0$$

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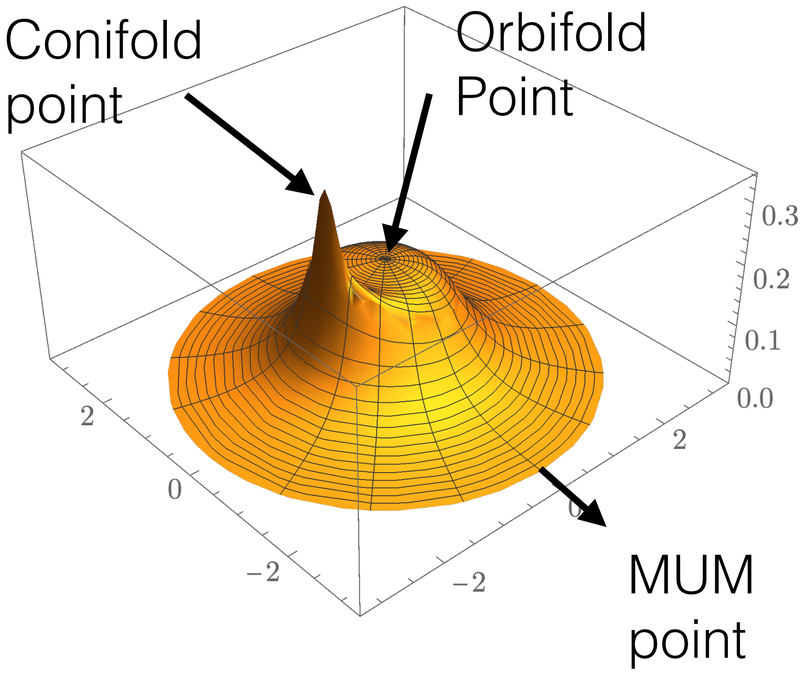

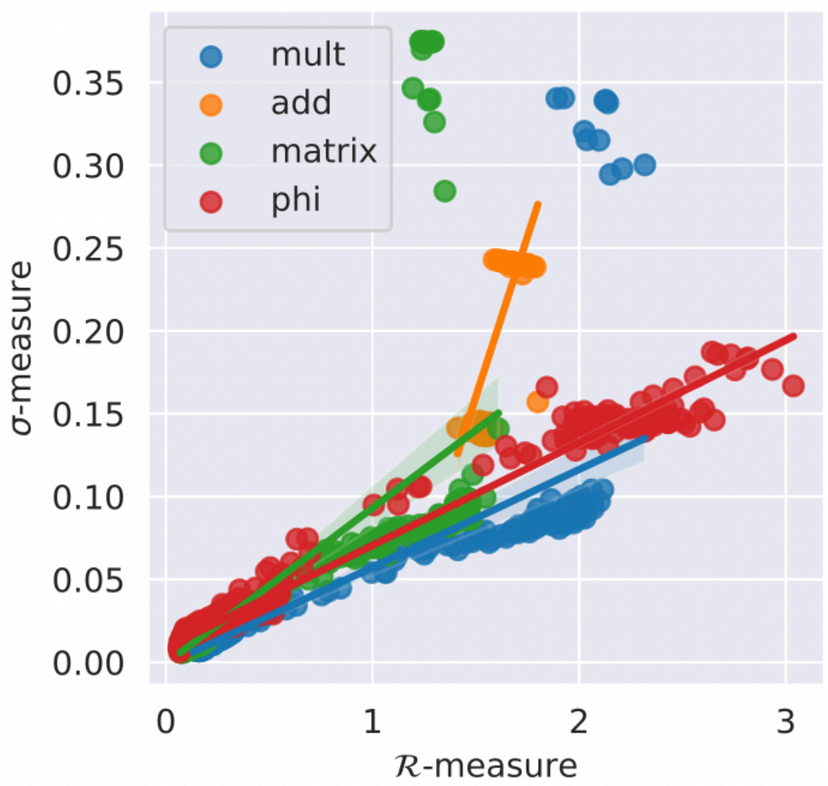

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$$\{\alpha_A\} = \frac{\left\{ s_\alpha^{(k_\phi)} \, \bar{s}_{\bar{\beta}}^{(k_\phi)} \right\}}{\left(|z_0|^2 + \ldots + |z_4|^2\right)^{k_\phi}}$$

w/ $s_\alpha^{(k_\phi)}$ degree k_ϕ monomials in z 's

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$\Delta_{AB} = \langle \alpha_A, \Delta\alpha_B \rangle, \quad O_{AB} = \langle \alpha_A, \alpha_B \rangle$ (evaluated using MC integration)

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- We compute at $k_\phi = 3 \Rightarrow$ can compute the first 1,225 eigenvalues
- For higher eigenvalues, lose accuracy due to finite truncation of basis

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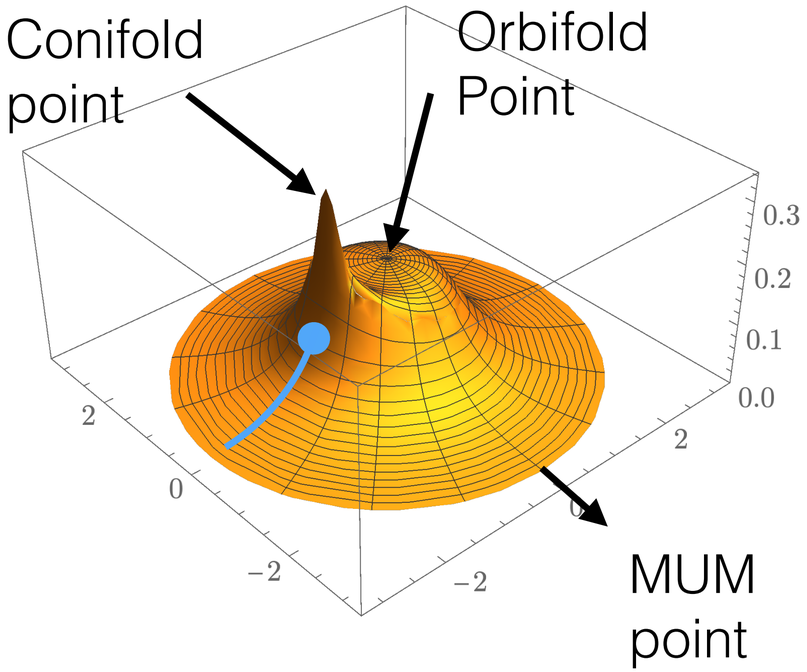
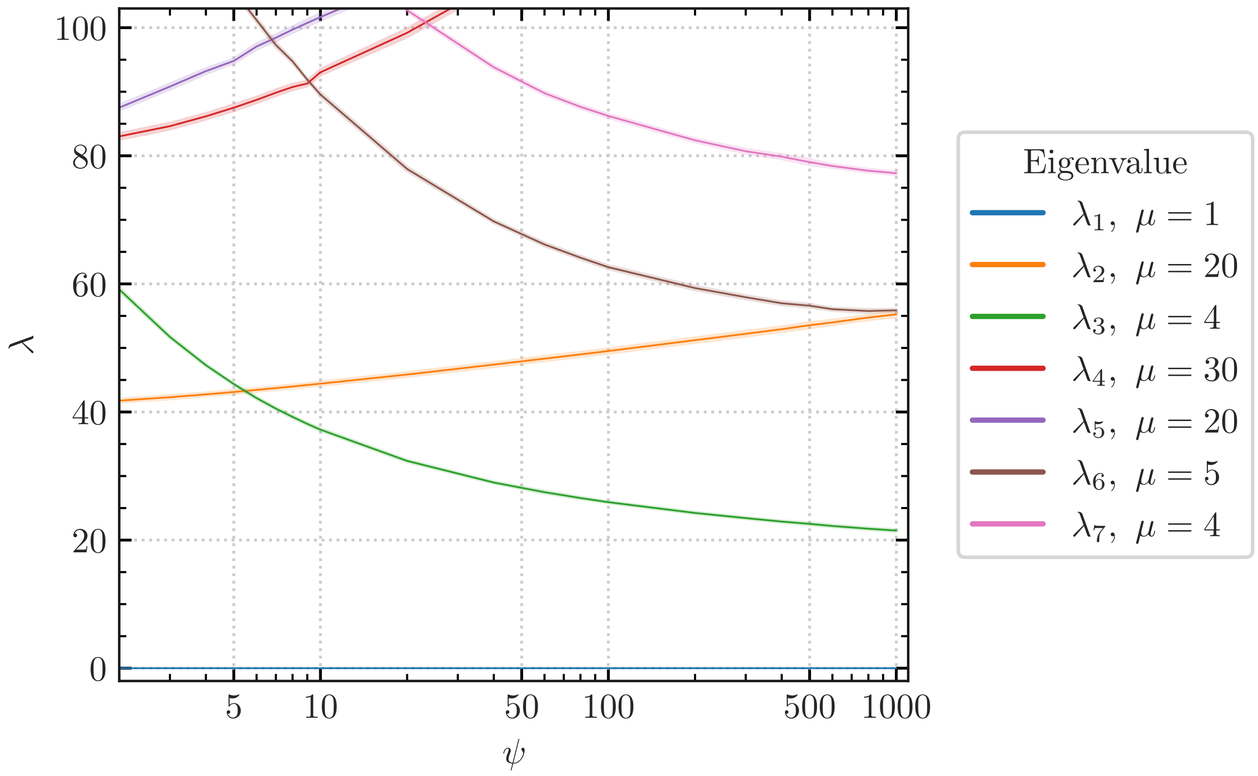
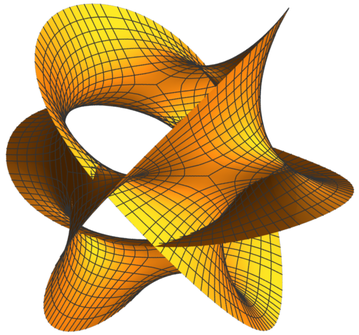


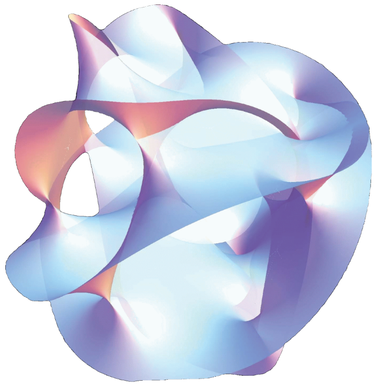
CS moduli space metric from solving the hypergeometric Picard-Fuchs system

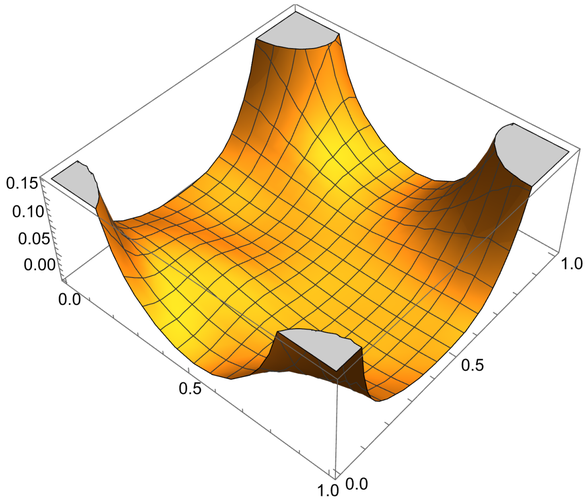


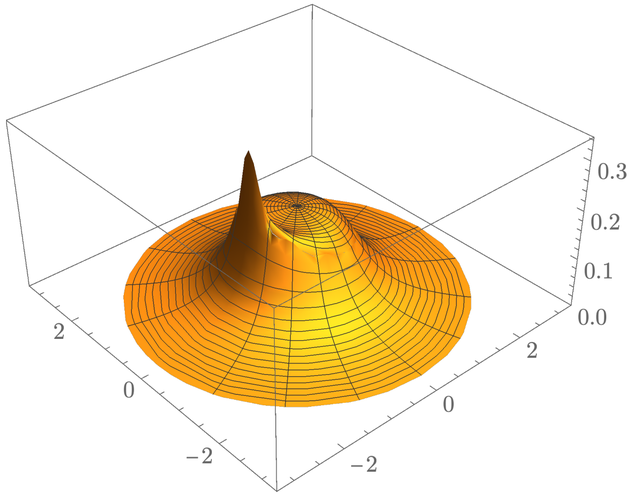
Change of scalar KK spectrum along geodesics in CS moduli space; degenerate in irreps of symmetry group of Quintic

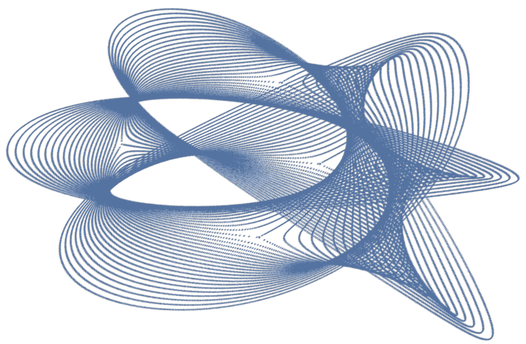
$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0 z_1 z_2 z_3 z_4 = 0$$

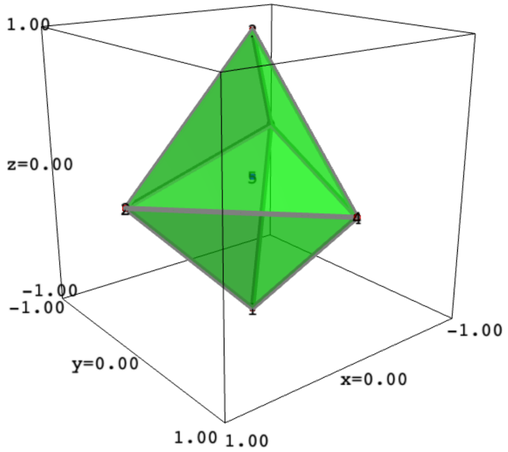


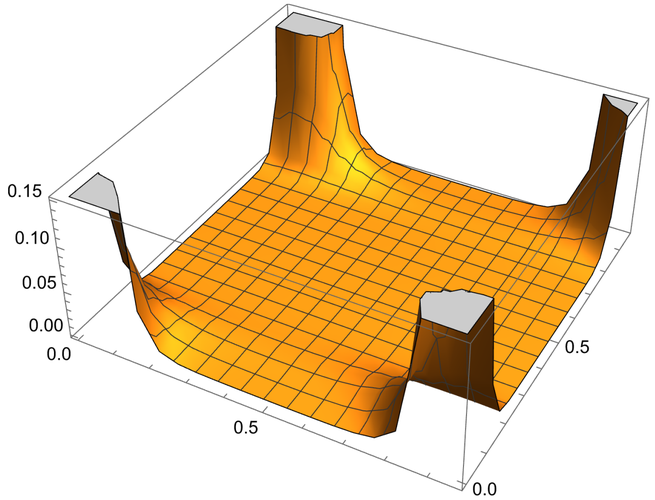


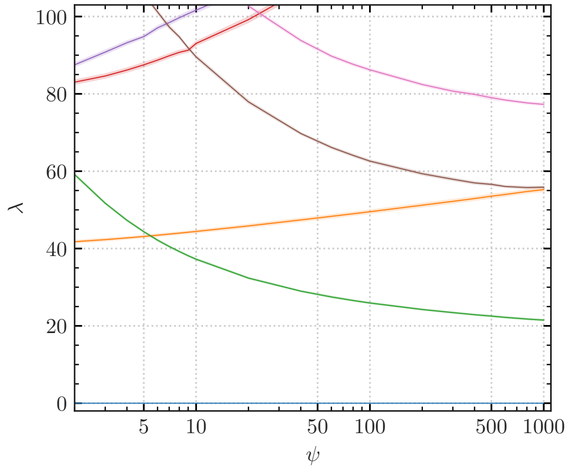












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  - Apply a theorem of Shiffman and Zelditch to find points on a CY with a known distribution
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- By adapting the loss that is optimized (e.g. relaxing $dJ = 0$), can find SU(n) / G_2 structure metrics [Larfors, Lukas, FR `18; Anderson, Gray, Gerdes, Krippendorf, Raghuram, FR: 2012.04656]
- Very good (Ricci-flat) approximations obtained within very short time
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Thank you for your attention!