Quantum duality and Morita theory for chiral algebras

Sam Raskin

University of Texas at Austin

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Overview

I want to discuss some joint work with Justin Hilburn, with an emphasis on formal patterns resembling parts of QFT.

These formal aspects have been influenced by many, many people (David Ben-Zvi, Sasha Braverman, Kevin Costello, Tudor Dimofte, Davide Gaiotto, Misha Finkelberg, and Philsang Yoo, for a start), and we do not claim any originality for it. I just wanted to take this opportunity today to do a little storytelling, recognizing that I picked up the ideas through osmosis while others did serious thinking around it.

Disclaimer

Speaking of osmosis: I am a geometric representation theorist, and do not have real training in math physics. I'm a tourist here with an outsider's motivations, and I may not be able to answer some simple questions that you ask!

Morita theory: motivation

In the physics literature, there is a remarkable role played by *dualities* of QFTs, many of which can ultimately be derived from a small list of string theory conjectures. Roughly speaking, one may find two QFTs Z_1 and Z_2, perhaps both with concrete Lagrangian/path-integral formulations, and one is told that the theories should be *fundamentally equivalent* in a way that does not at all respect their Lagrangian presentations. This means that fields in the two theories take different forms (e.g., one may be a gauge theory, the other not), but some *fundamental physical features* match.

Morita theory: motivation

I wish to take the following as an ansatz.

Many theories are defined by a (factorization) algebra $\mathcal{A}$ of local operators. Others are not too far from such a case: e.g., they may be "non-affine" generalizations of a family of such examples.

In this case, equivalence/duality of QFTs $Z_{\mathcal{A}}$ and $Z_{\mathcal{B}}$ should amount to a *Morita equivalence* between $\mathcal{A}$ and $\mathcal{B}$.

We will discuss the $1d$ ($2d$?) case as a warm up, and then specialize to the $2d$ ($3d$?) algebraic/holomorphic setting.

Morita theory: background

Suppose A and B are associative algebras. A *Morita equivalence* between them is an equivalence A–mod $\simeq$ B–mod.

Technical comment: my categories are all derived, which will matter at certain points. So this is often called *derived* Morita equivalence.

Such a *categorical* relationship between A and B leads to concrete relationships between A and B. Famously: $HH_\bullet(A) \simeq HH_\bullet(B)$ as chain complexes with S^1-actions.

More generally, I think of Morita theory as considering the *category* A–mod rather than the algebra A itself. This philosophy is also sometimes called *non-commutative geometry*, and surely it goes by other names as well.

Morita theory: TFT

One can reinterpret the above as follows. First, an associative algebra A determines a (dualizable) category A–mod.

More generally, any (*dualizable*) DG category $\mathcal{C}$ essentially determines a $2d$ (extended) TFT $Z_{\mathcal{C}}$. Here *essentially* properly means $[0, 1]$-extended, i.e., defined on 0 and 1-dimensional manifolds.

Morita theory from the TFT POV

Explicitly, the theory $Z_{\mathcal{C}}$ assigns:

- The category $\mathcal{C}$ to the (framed) point.
- The (chain complex of) vector space(s) $HH_\bullet(\mathcal{C})$ to the circle S^1.

Remark

In the case $\mathcal{C} = A$–mod (breaking Morita invariance!), an n-tuple of points $\{x_1, \ldots, x_n\} \subseteq S^1$ induces a natural map $A^{\otimes n} \to HH_\bullet(A) = HH_\bullet(A\text{–mod})$, which one can think of as analogous to correlators between local operators (if we pretend $HH_\bullet(A) = \mathbb{C}$, or at least suppose we are given a "path integral" map $HH_\bullet(A) \to \mathbb{C}$).

Remark

The theory $Z_{\mathcal{C}}$ extends to framed 2-manifolds when $\mathcal{C}$ is smooth and proper. In this case, the theory attaches $\dim(HH_\bullet(\mathcal{C}))$ to $\mathbb{T}^2$.

An aside about dimensions

Question: we think of A as defining quantum mechanics, not a $2d$ theory. What is happening?

Properly, in many circumstances, a physicist's n-dimensional QFT comes in a *pair*: an n-dimensional theory of *states*, and an $(n+1)$-dimensional theory of *observables*. The state theory is naturally a boundary condition for the theory of observables.

In the above example, we think of the "Hilbert space" V as the theory of states, the $2d$ theory Z_A as the theory of observables, and the action of A on V as encoded by realizing $V \in A$–mod.

In this talk, I'm more interested in theories of observables, so sometimes the dimensions may seem off.

Chiral algebras

The theory of chiral (alias: factorization) algebras was developed by Beilinson-Drinfeld. I want to give a super brief recap about the outcomes of their theory.

The setting is a smooth algebraic curve X over a field k (say, $k = \mathbb{C}$) of characteristic 0. In this setting, one may speak of *chiral algebras* $\mathcal{A}$ on X. Vertex algebras are roughly equivalent to chiral algebras defined on *every* curve, and most chiral algebras in nature arise by this procedure.

We will discuss some examples later in the talk, but not yet.

Chiral algebras: local structure

For a chiral algebra $\mathcal{A}$ on X and a fixed point $x \in X$, there is a category $\mathcal{A}\text{–mod}_x$ of *chiral* $\mathcal{A}$*-modules supported at* x.

Typically, $\mathcal{A}\text{–mod}_x$ admits a concrete description involving Laurent series $k((t))$, which geometrically corresponds to the punctured disc $\overset{o}{\mathcal{D}}_x$. For example, $\mathfrak{g}((t))\text{–mod}$ is the category of modules for the (central charge 0) Kac-Moody chiral algebra.

Example

There is always a canonical *vacuum* chiral module $\mathcal{A}_x \in \mathcal{A}\text{–mod}_x$. It is more analogous to A considered as an A-*bi*module than to A considered as an A-module.

Chiral algebras: global structure

Now suppose that X is *projective*.

Then there is a vector space $H_{ch}(X, \mathcal{A})$ of *chiral homology/conformal blocks* for the theory. For any n-tuple of points $\{x_1, \ldots, x_n\} \in X$, there is a map:

$$\mathcal{A}_{x_1} \otimes \ldots \otimes \mathcal{A}_{x_n} \to H_{ch}(X, \mathcal{A})$$

that again roughly encodes correlators.

Typically, $H_{ch}(X, \mathcal{A})$ is computed using geometry of the projective curve X.

Chiral algebras: global structure

More generally, there is a *chiral homology* functor:

$$\mathcal{A}\text{–mod}_{x_1} \otimes \ldots \otimes \mathcal{A}\text{–mod}_{x_n} \to \mathsf{Vect}$$
$$M_1, \ldots, M_n \mapsto H_{ch}(X, \mathcal{A}; M_1, \ldots, M_n)$$

and natural maps:

$$M_1 \otimes \ldots \otimes M_n \to H_{ch}(X, \mathcal{A}; M_1, \ldots, M_n).$$

Inserting the vacuum module at a point does not change the chiral homology.

Algebraic field theories

Thesis: the Morita theory of chiral algebras is governed by 3*d* *algebraic field theories*.

Algebraic field theories

Recall that $3d$ TFTs assign numbers to closed 3-manifolds, vector spaces to 2-manifolds, and categories to 1-manifolds.

We have the following (outline of a) definition.
An (algebraic, [1,2]-extended) *3d field theory* Z on a smooth, proper algebraic curve X is ... some data.

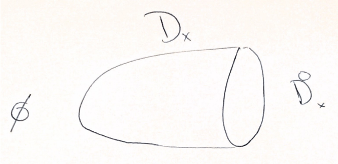
Algebraic field theories

First, for $x \in X$, we have a DG category $Z(\overset{o}{\mathcal{D}}_x) \in \mathrm{DGCat}$ – roughly, this is the category attached to the "1-manifold/circle" $\overset{o}{\mathcal{D}}_x$.

$Z(\overset{o}{\mathcal{D}}_x)$ is usually called the *category of line operators* for the theory Z.

Algebraic field theories

The "cobordism:"



defines a *vacuum* (or *unit*) object:

$$\mathbb{1}_x \in Z(\mathring{\mathcal{D}}_x).$$

Remark

Something funny: in the algebraic setting, cobordisms are inherently asymmetric. For me, this is just an experimental fact.

Algebraic field theories

The "cobordism":

defines a *chiral homology* functor:

$$H_{ch,x}(-) : Z(\mathring{\mathcal{D}}_x) \to Z(\varnothing) = \text{Vect}.$$

Algebraic field theories

The vector space $H_{ch,x}(\mathbb{1}_x) \in \mathsf{Vect}$ is independent of the choice of point x, and should be thought of as the vector space the theory assigns to the "2-manifold" X.

Algebraic field theories

There are other (more technical) axioms about varying points and vacuum insertion, but this is the top line stuff.

Fantasy

What is a $[0, 2]$-extended algebraic field theory?

Algebraic field theories

Main example: given a chiral algebra $\mathcal{A}$ on X, Beilinson-Drinfeld defined a $3d$ theory $Z_{\mathcal{A}}$.

Here $Z_{\mathcal{A}}(\overset{o}{\mathcal{D}}_x)$ is the category of $\mathcal{A}$-modules supported at x. The vacuum representation of $\mathcal{A}$ is the unit object, and chiral homology is given as defined by Beilinson-Drinfeld.

One can take *Morita theory for chiral algebras* to mean equivalences of theories attached to chiral algebras. Justin Hilburn and I proved something non-trivial of this shape (except one side is a little non-affine), which I want to explain now.

Where do Morita equivalences of chiral algebras come from?

Answer: there are many precise predictions coming from the $3d$ *mirror symmetry program*.

Attributions

The circle of ideas I'm going to discuss was developed by many people. I apologize for any incompleteness and inaccuracies here. On the mathematical physics side, we are discussing $3d$ mirror symmetry (and implicitly, its relation with $4d$ S-duality). The former subject was initiated by Intriligator–Seiberg and further developed by Hanany–Witten and many others. The connections with S-duality were first considered by Gaiotto–Witten. The sharp mathematical conjectures were first considered by Hilburn-Yoo (maybe jointly with Dimofte-Gaitto) and later Braverman–Finkelberg.

Important parts of the story have cousins in harmonic analysis. This body of conjectures originated from work of Sakellaridis–Venkatesh that provides a unified perspective on period expressions for L-functions.

Forthcoming work of Ben-Zvi–Sakellaridis–Venkatesh joins these two perspectives and connects with the geometric Langlands program.

Lagrangian $3d$ $\mathcal{N} = 4$ theories

For a stack Y, physicists say that the (non-algebraic!) $3d$ σ-model Z_Y with target T^*Y has $\mathcal{N} = 4$ supersymmetry. This allows us to define two *twists* of Z_Y, the A and B-twists. In practice, these $3d$ theories $Z_{Y,A}$ and $Z_{Y,B}$ are *algebraic*, and I will treat them as such.

Lagrangian $3d$ $\mathcal{N} = 4$ theories

The basic properties of these twists are:

$$Z_{Y,A}(\overset{o}{\mathcal{D}}_x) = D(\mathcal{M}\text{aps}(\overset{o}{\mathcal{D}}_x, Y))$$
$$Z_{Y,B}(\overset{o}{\mathcal{D}}_x) = \text{IndCoh}(\mathcal{M}\text{aps}(\overset{o}{\mathcal{D}}_{x,dR}, Y)).$$

Here D denotes the category of D-modules on this (infinite dimensional!) space.

Remark

For Y affine, $D(\mathcal{M}\text{aps}(\overset{o}{\mathcal{D}}_x, Y)$ is the category of modules for a CDO (alias: curved $\beta\gamma$-system) attached to Y, while $\text{IndCoh}(\mathcal{M}\text{aps}(\overset{o}{\mathcal{D}}_{x,dR}, Y)$ is modules for a constant commutative chiral algebra with fiber $\text{Fun}(Y)$.

Mirror symmetry

For a $3d$ $\mathcal{N} = 4$ theory Z, there is a *abstract mirror dual theory* $Z^\star$. This should be the same $3d$ theory but with the supersymmetries realized in a conjugate way. By construction, $Z_A = Z_B^\star$ and $Z_B = Z_A^\star$.

Mirror symmetry (in 3-dimensions) refers to *mirror dual* pairs (Y_1, Y_2) with $Z_{Y_1} = Z_{Y_2}^\star$. Given the previous slide, any such pair (Y_1, Y_2) yield interesting mathematical conjectures.

Mirror symmetry

There are many (expected) examples of mirror dual pairs, and I do not intend to survey them here.

But let me give one example that is important to me: $Y_1 = \mathbb{A}^1$ and $Y_2 = \mathbb{A}^1/\mathbb{G}_m$. The theory Z_{Y_1} is that of a pure hypermultiplet, while the theory Z_{Y_2} is that of a $U(1)$-gauged hypermultiplet.

Here the prediction $Z_{Y_1,A}(\overset{o}{\mathcal{D}}_x) \simeq Z_{Y_2,B}(\overset{o}{\mathcal{D}}_x)$ amounts to an equivalence:

$$\mathrm{IndCoh}(\mathcal{M}\mathrm{aps}(\overset{o}{\mathcal{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m)) \simeq D(\mathcal{M}\mathrm{aps}(\overset{o}{\mathcal{D}}, \mathbb{A}^1)).$$

This is my theorem with Hilburn. (Technically, this is just the category of line operators. We worked out but have not written up the global aspects yet.)

As far as I know, this is the first honestly derived Morita theorem for chiral algebras (or near enough), and the first non-classical example.

Geometric side

The me in the past who wrote this talk does not think there is time to say much more, but just in case, let's discuss the geometry of both sides a little more. Below I'll let $O = k[[t]]$ and let $K = k((t))$.

For Y an affine scheme, I let $\mathfrak{L}^+ Y = \mathcal{M}\text{aps}(\mathcal{D}_x, Y)$ (resp. $\mathfrak{L}Y = \mathcal{M}\text{aps}(\overset{o}{\mathcal{D}}_x, Y)$) denote the scheme (resp. ind-scheme) such that:

$$\{\text{Spec}(A) \to Y(O)\} = \{\text{Spec}(A[[t]]) \to Y\}$$
$$\{\text{Spec}(A) \to Y(K)\} = \{\text{Spec}(A((t))) \to Y\}.$$

Geometric side

There are theories of D-modules on such spaces.

For example, I can think of $K = \mathfrak{L}\mathbb{A}^1$ as an (ind-pro) affine space with coordinates a_i given by Laurent coefficients. Then a *quasi-coherent sheaf* on K is a vector space V with operators $a_i : V \to V$ ($i \in \mathbb{Z}$) such that for any vector $v \in V$, $a_i v = 0$ for $i \ll 0$. A *D-module* on K also has operators ∂_{a_i} such that for every vector v, $\partial_{a_i} v = 0$ for $i \gg 0$ and such that $[\partial_{a_i}, a_j] = \delta_{ij} \cdot \mathrm{id}$.

Such data are equivalent to modules over the Weyl (or $\beta\gamma$, or CDO on $\mathbb{A}^1$) VOA.

Geometric side

The above definition has a more categorical expression:

$$D(K) = \operatorname{colim}_n D(t^{-n}O) = \operatorname{colim}_n \operatorname{colim}_m D(t^{-n}O/t^m O).$$

This formula means that $D(K)$ is built out of categories of D-modules on finite-dimensional affine spaces $t^{-n}O/t^m O$ via D-module functoriality.

This definition is well-behaved on derived categories, and we take it as our definition of $D(\mathfrak{L}\mathbb{A}^1)$.

Geometric side

We similarly have $D(\mathfrak{L}\mathbb{G}_m)$, which is a monoidal category under convolution. It acts canonically on $D(K)$.

From the perspective of geometric representation theory, our problem is to understand the spectral decomposition of $D(K)$ as a $D(\mathfrak{L}\mathbb{G}_m)$-module category.

Spectral side

For this problem to make sense, we recall local geometric class field theory:

Theorem (Beilinson-Drinfeld)

There is a canonical equivalence $D(\mathfrak{L}\mathbb{G}_m) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m})$ *as symmetric monoidal categories.*

Here $\mathrm{LocSys}_{\mathbb{G}_m}$ is the moduli of rank 1 de Rham local systems $(\mathcal{L}, \nabla)$ on $\overset{o}{\mathcal{D}} := \mathrm{Spec}(K)$. By definition, we take $\mathfrak{L}\mathbb{G}_m$ acting on $\mathfrak{L}\mathbb{A}^1 dt$ via the homomorphism $d\log : \mathfrak{L}\mathbb{G}_m \to \mathfrak{L}\mathbb{A}^1 dt$ and form the (stack) quotient $\mathrm{LocSys}_{\mathbb{G}_m}$.

Spectral side

Now define $\mathcal{Y}$ as the moduli of data $(\mathcal{L}, \nabla, s)$ with $(\mathcal{L}, \nabla)$ a rank 1 local system on the punctured disc and $s \in \mathcal{L}$ with $\nabla(s) = 0$. I.e., $\mathcal{Y} = \mathcal{M}\mathrm{aps}(\overset{o}{\mathcal{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m)$.

At a technical level, the structure of $\mathcal{Y}$ is given by:

Proposition

$\mathcal{Y}$ is a quotient of a classical ind-affine scheme by an action of $\mathfrak{L}\mathbb{G}_m/(1 + tO)$.

As a consequence of this result, we can make sense of $\mathrm{IndCoh}(\mathcal{Y})$.

Remark

I found understanding the geometry of $\mathcal{Y}$ to be the most challenging part of our work.

Statement of the main result, redux

Theorem (Hilburn-R.)

There is a canonical equivalence of DG categories:

$$\mathsf{IndCoh}(\mathcal{Y}) \simeq D(\mathfrak{L}\mathbb{A}^1)$$

compatible with local geometric class field theory.

Remark

Physically, the compatibility with class field theory is expressed as compatibility with the structure of the $3d$ theory as boundary conditions for S-dual abelian Yang-Mills in $4d$.

Remark

If $(\mathcal{L}, \nabla)$ is a non-trivial local system with connection, the fiber of $\mathcal{Y} \to \mathsf{LocSys}_{\mathbb{G}_m}$ over it is just a point. This is similar to phenomena from Tate's thesis.

Thanks!