BPS Dendroscopy on Local Calabi-Yau Threefolds

Boris Pioline

String-Math 2022, 14/07/2022

Work in progress with Pierrick Bousseau, Pierre Descombes and Bruno Le Floch
δέντρον = tree

Dentrology

Dendrochronology
δενδροσκοπία = analyzing the BPS spectrum in terms of attractor flow trees

Attractor flow trees on $K_{\mathbb{P}^2}$, $\gamma = [1, 0, -3]$, $\mathcal{M} = \text{Hilb}_4\mathbb{P}^2$
In type IIA string theory compactified on a Calabi-Yau threefold $X$, the BPS spectrum consists of bound states of D6-D4-D2-D0 branes, described mathematically by objects $E$ in the derived category of coherent sheaves $\mathcal{C} = D^b\text{Coh}(X)$ [Douglas'01].

The BPS index or Donaldson-Thomas invariant $\Omega(z(\gamma))$ counts stable states with charge $\gamma = \text{ch}(E) \in H^{\text{even}}(X, \mathbb{Q})$ saturating the BPS bound $M(\gamma) \geq |Z(\gamma)|$, where $Z \in \text{Hom}(\Gamma, \mathcal{C})$ depends on the complexified Kähler moduli $z \in M$.

$\Omega(z(\gamma))$ is locally constant on $M$, but can jump across real codimension one walls of marginal stability $W(\gamma_1, \gamma_2) \subset M$, where the phases of the central charges $Z(\gamma_1)$ and $Z(\gamma_2)$ with $\gamma = m_1\gamma_1 + m_2\gamma_2$ become aligned [Kontsevich Soibelman'08, Joyce Song'08].

Physically, multi-centered black hole solutions (dis)appear across the wall [Denef Moore '07, ..., Manschot BP Sen '11].
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Introduction

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- Physically, multi-centered black hole solutions (dis)appear across the wall [Denef Moore ’07, …, Manschot BP Sen ’11].
For a non-compact CY3 of the form $X = K_S$ where $S$ is a complex Fano surface, there is an injection $\iota_* : D^b\text{Coh}(S) \rightarrow D^b_c(X)$ lifting an object $E$ with Chern character $\gamma = [r, d, \text{ch}_2]$ to a bound state of $r$ D4-branes wrapped on $S$, $c$ D2-branes and $\text{ch}_2$ D0-branes.

At large volume, the central charge is quadratic in complexified Kähler moduli $z_a = b_a + i t_a$, $Z(\gamma) \sim -\int_S e^{-z_a H_a} \text{ch} E = -r z_a Q_{ab} z_b + z_a d_a - \text{ch}_2 \Omega z(\gamma)$ reduces to the Gieseker index $\Omega_\infty(\gamma)$, given (up to sign) by the Euler number of the moduli space of Gieseker semi-stable sheaves on $S$ with Chern character $\gamma$.

At finite volume, $Z$ receives worldsheet instanton corrections computable by mirror symmetry. Can we determine $\Omega z(\gamma)$ anywhere, and understand what are BPS states really “made of”? 

B. Pioline (LPTHE, Paris)
BPS spectrum on local surfaces

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Outline

1. Introduction
2. Kähler moduli space of $K_{\mathbb{P}^2}$
3. Attractor flow trees and scattering diagrams
4. Large volume scattering diagram
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2. Kähler moduli space of $K_{\mathbb{P}^2}$

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5. Towards the exact scattering diagram
The Kähler moduli space of $K_{\mathbb{P}^2}$ is the modular curve $X_1(3) = \mathbb{H}/\Gamma_1(3)$ parametrizing elliptic curves with level structure. It admits two cusps $L V, C$ and one elliptic point $o$ of order 3.
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The universal cover is parametrized by $\tau \in \mathbb{H}$:

$$Z_{\tau}(\gamma) = -rT_D(\tau) + dT(\tau) - \text{ch}_2$$

$$T = \int_\ell \lambda$$

$$T_D = \int_{\ell_D} \lambda$$

$\lambda$ holomorphic one-form with logarithmic singularities on $\mathcal{E}_\tau$
Central charge as Eichler integral

Since $\partial_{\tau} \lambda$ is holomorphic, its periods are proportional to $(1, \tau)$. Integrating on a path from $o$ to $\tau$, one finds the Eichler-type integral

$$\begin{pmatrix} T \\ T_D \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} + \int_{\tau_o}^{\tau} \begin{pmatrix} 1 \\ u \end{pmatrix} C(u) \, du$$

where $C(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3}$ is a weight 3 modular form for $\Gamma_1(3)$. 

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- This provides an computationally efficient analytic continuation of $Z_\tau$ throughout $\mathbb{H}$, and gives access to monodromies:

\[
\tau \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & d & c \\ m_D & b & a \end{pmatrix} \cdot \begin{pmatrix} 1 \\ T \\ T_D \end{pmatrix}
\]

where $(m, m_D)$ are period integrals of $C$ from $\tau_o$ to $\frac{a\tau_o - b}{c\tau_o - d}$.
Central charge as Eichler integral

At large volume, using \( C = 1 - 9q + \ldots \) one finds

\[
T = \tau + \mathcal{O}(q), \quad T_D = \frac{1}{2} \tau^2 + \frac{1}{8} + \mathcal{O}(q)
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For \( \tau_2 \) large enough, one can use the \( GL(2, \mathbb{R})^+ \) action on space of Bridgeland stability conditions to absorb the \( \mathcal{O}(q) \) corrections:

\[
Z_{LV}^{(s,t)}(\gamma) = -\frac{r}{2} (s + it)^2 + d(s + it) - \text{ch}_2,
\]

\[
s = \frac{\text{Im}T_D}{\text{Im}T}, \quad \mu = \frac{d}{r}
\]

\[
\frac{1}{2} (s^2 + t^2) = -\frac{\text{Im}(T \bar{T}_D)}{\text{Im}T}
\]

\[
\mathcal{A} = \{ E \xrightarrow{d} F, \mu(E) < s, \mu(F) \geq s \}
\]

[Bayer Macri’11]
Near the orbifold point $\tau_o = -\frac{1}{2} + \frac{i}{2\sqrt{3}}$, the BPS spectrum is governed by a quiver with potential:

\[ W = \sum \epsilon_{ijk} X_i Y_j Z_k \]

- $E_1 = \mathcal{O}$  \quad $\gamma_1 = [1, 0, 0]$
- $E_2 = \Omega(1)[1]$,  \quad $\gamma_2 = [-2, 1, \frac{1}{2}]$
- $E_3 = \mathcal{O}(-1)[2]$,  \quad $\gamma_3 = [1, -1, \frac{1}{2}]$

\[
\begin{align*}
r & = 2n_2 - n_1 - n_3 \\
d & = n_3 - n_2 \\
\text{ch}_2 & = -\frac{1}{2}(n_2 + n_3)
\end{align*}
\]

The BPS index $\Omega_{\tau}(\gamma)$ coincides with the (signed) Euler number $\Omega_{\zeta}(\gamma)$ of the moduli space of King semi-stable representations of dimension $\gamma = (n_1, n_2, n_3)$, with FI-parameters $\theta_i = \text{Im}Z_\tau(\gamma_i)$. 
Attractor conjecture for $K_{\mathbb{P}^2}$

In the chamber $\theta_1 > 0, \theta_3 < 0$, the arrows $Z_k$ vanish in any stable representation, and $(Q, W)$ reduces to the Beilinson quiver describing normalized torsion-free sheaves on $\mathbb{P}^2$:

$$
n_1 \xrightarrow{X_i} n_2 \xrightarrow{Y_j} n_3 \quad \epsilon_{ijk} X^i Y^j = 0$$

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In [Beaujard BP Manschot’20], we showed that the attractor index $\Omega_\star(\gamma) := \Omega_{\langle \gamma, - \rangle}(\gamma)$ vanishes except for $\gamma = \gamma_i$ or $\gamma \propto \gamma_1 + \gamma_2 + \gamma_3$. 
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\begin{align*}
\begin{array}{ccc}
\circ & \xrightarrow{X_i} & \circ \\
\downarrow & & \downarrow \\
n_1 & & n_2 \\
\end{array}
\begin{array}{ccc}
\circ & \xrightarrow{Y_j} & \circ \\
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\end{align*}

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Moreover, the anti-attractor index $\Omega_x (\gamma) := \Omega_{-(\gamma, -)} (\gamma)$ coincide with the Gieseker index $\Omega_{\infty} (\gamma)$, provided $-r < d \leq 0$. 
Attractor conjecture for $K_{\mathbb{P}^2}$

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Moreover, the anti-attractor index $\Omega_x(\gamma) := \Omega_{-(\gamma,-)}(\gamma)$ coincide with the Gieseker index $\Omega_\infty(\gamma)$, provided $-r < d \leq 0$.

A similar conjecture for $\Omega^*(\gamma)$ holds for any toric CY3, giving in principle access to DT invariants $\Omega_\zeta(\gamma)$ for any $\zeta \in \mathbb{R}^{Q_0}$ [Mozgovoy BP’20; Descombes’21]
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The Attractor Flow Tree Formula expresses the BPS index $\Omega_\theta(\gamma)$ for any (generic) $\theta \in \mathbb{R}^{Q_0}$ in terms of attractor indices by summing over all possible flow trees: schematically,

$$
\Omega_\theta(\gamma) \sim \sum_{\gamma=\gamma_1+\cdots+\gamma_n} \left( \sum_{T \in T_\theta(\{\gamma_i\})} \prod_{v \in V_T} \langle \gamma_L(v), \gamma_R(v) \rangle \right)^n \prod_{i=1}^n \Omega_*(\gamma_i)
$$

*Denef ’00; Denef Greene Raugas ’01; Denef Moore’07; Manschot ’10, Alexandrov BP’18*
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Here, a flow tree $T$ is a binary rooted tree, with edges decorated with charges $\gamma_e$, such that $\gamma_v = \gamma_L(v) + \gamma_R(v)$ at each vertex, with charges $\gamma_i$ assigned to the leaves and $\gamma$ to the root.
The Attractor Flow Tree Formula for quivers

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- Each edge is embedded in $\mathbb{R}^{Q_0}$ along $\theta_v = \theta_{p(v)} + \lambda \langle \gamma_e, - \rangle$, $\lambda \geq 0$, such that the root vertex maps to $\theta$, and $(\theta_v, \gamma_L(v)) = (\theta_v, \gamma_R(v)) = 0$ at each vertex.
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Split attractor flows

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The physical picture is that typical multi-centered solutions in $\mathcal{N} = 2$ supergravity have a nested structure. The linear flow in $\theta$ originates from gradient flow for spherically symmetric black holes [Ferrara Kallosh Strominger'95].

At each level $\nu$, the average distance between the clusters of charge $\gamma_L(\nu)$ and $\gamma_R(\nu)$ is fixed, but the orientation in $S^2$ gives $|\langle \gamma_L(\nu), \gamma_R(\nu) \rangle|$ degrees of freedom. In addition, each center of charge $\gamma_i$ carries internal degrees of freedom counted by $\Omega_* (\gamma_i)$. 

\[ r^2 \frac{dz^i}{dr} = -g^{ij} \partial_j |Z(\gamma)|^2 \]
In order to enforce Bose-Fermi statistics whenever two charges coincide, one should replace $\Omega_\theta(\gamma)$ by the rational index $\bar{\Omega}_\theta(\gamma) = \sum_{d|\gamma} \frac{1}{d^2} \Omega_\theta\left(\frac{\gamma}{d}\right)$ and insert a Boltzmann symmetry factor.

[Manschot BP Sen’11]
In order to enforce **Bose-Fermi statistics** whenever two charges coincide, one should replace \( \Omega_\theta(\gamma) \) by the rational index
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\tilde{\Omega}_\theta(\gamma) = \sum_{d \mid \gamma} \frac{1}{d^2} \Omega_\theta(\gamma_d)
\]
and insert a Boltzmann symmetry factor.

[Manachot BP Sen’11]

When the charges \( \gamma_i \) are not linearly independent, some splittings can involve higher valency vertices. One can treat them using the full KS wall-crossing formula, or perturb \( \theta \) such that only binary trees remain.
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The attractor flow tree formula is consistent with wall-crossing: the index jumps when $z$ crosses the wall $\mathcal{W}(\gamma_L(v_0), \gamma_R(v_0))$ associated to the primary splitting for one of the trees.
There are additional ‘fake walls’ where the topology of the trees jump but the total index is constant, thanks to the identity

\[ \langle \gamma_1, \gamma_2 \rangle \langle \gamma_1 + \gamma_2, \gamma_3 \rangle + \text{cyc.} = 0 \]
Remarks

- There are additional ‘fake walls’ where the topology of the trees jump but the total index is constant, thanks to the identity

\[
\langle \gamma_1, \gamma_2 \rangle \langle \gamma_1 + \gamma_2, \gamma_3 \rangle \text{ + cyc.} = 0
\]

- The formula can be refined by replacing

\[
\langle \gamma_L, \gamma_R \rangle \rightarrow \frac{y^{\langle \gamma_L, \gamma_R \rangle} - y^{-\langle \gamma_L, \gamma_R \rangle}}{y - 1/y}
\]

\[
\bar{\Omega}_\theta(\gamma) \rightarrow \bar{\Omega}_\theta(\gamma, y) = \sum_{d \mid \gamma} \frac{y^{-1/y}}{d(y^d - y^{-d})} \Omega_\theta(\gamma \div d, y^d)
\]

Physically, \( y \) is a fugacity conjugate to angular momentum in \( \mathbb{R}^3 \).
Flow tree formula from scattering diagrams

For any quiver with potential \((Q, W)\), the scattering diagram \(D\) is the set of real codimension-one rays \(\{\mathcal{R}(\gamma), \gamma \in \mathbb{Z}^{Q_0}\}\) defined by [Bridgeland'16]

\[
\mathcal{R}(\gamma) = \{\zeta \in \mathbb{R}^{Q_0} : (\zeta, \gamma) = 0, \Omega_{\zeta}(k\gamma) \neq 0 \text{ for some } k \geq 1\}
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- Each point along \(\mathcal{R}(\gamma)\) is endowed with an automorphism of the quantum torus algebra, (assume \(\gamma\) primitive)

\[
\mathcal{U}(\gamma) = \exp\left(\sum_{m=1}^{\infty} \frac{\Omega_{\zeta}(k\gamma,y)}{y^{-1}-y} X_{k\gamma}\right), \quad X_\gamma X_\gamma' = (-y)^{\langle \gamma, \gamma' \rangle} X_{\gamma+\gamma'}
\]

- The WCF ensures that the diagram is consistent, \(\prod_{\gamma_i} \mathcal{U}(\gamma_i)^{\pm 1} = 1\) around any codimension 2 intersection. The Attractor Flow Tree Formula determines outgoing rays from incoming rays at each vertex. [Argüz Bousseau '20].
A 2D slice of the orbifold scattering diagram

\[ \gamma_1 + \gamma_2 \]
\[ 2\gamma_1 + \gamma_2 \]
\[ \gamma_3 + \gamma_1 \]
\[ \gamma_3 + 2\gamma_1 \]
\[ 2\gamma_2 + \gamma_3 \]
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\[ \gamma_1 \]
\[ \gamma_2 \]
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B. Pioline (LPTHE, Paris)
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BPS Dendroscopy
More generally, for any $\psi \in \mathbb{R}/2\pi \mathbb{Z}$ define scattering rays as codimension-one loci in the space of Bridgeland stability conditions

$$\mathcal{R}_\psi(\gamma) = \{ Z : \text{Re}(e^{-i\psi} Z(\gamma)) = 0, \text{Im}(e^{-i\psi} Z(\gamma)) > 0, \Omega_\zeta(\kappa \gamma) \neq 0 \}$$
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For a non-compact CY3, $Z(\gamma)$ is holomorphic in Kähler moduli, thus $\arg Z(\gamma)$ is constant along the gradient flow of $|Z(\gamma)|$. Choosing $\psi$ such that $z \in R_\psi(\gamma)$, edges of attractor flow trees lie inside $R_\psi(\gamma_e)$, while vertices lie in $R_\psi(\gamma_L(v)) \cap R_\psi(\gamma_R(v))$. 
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- Besides, since $Z(\gamma)$ is holomorphic, initial rays must originate from attractor points on the boundary.
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$$\mathcal{R}_\psi(\gamma) = \{ Z : \text{Re}(e^{-i\psi} Z(\gamma)) = 0, \text{Im}(e^{-i\psi} Z(\gamma)) > 0, \Omega_\zeta(k\gamma) \neq 0 \}$$

For a non-compact CY3, $Z(\gamma)$ is holomorphic in Kähler moduli, thus $\arg Z(\gamma)$ is constant along the gradient flow of $|Z(\gamma)|$. Choosing $\psi$ such that $z \in \mathcal{R}_\psi(\gamma)$, edges of attractor flow trees lie inside $\mathcal{R}_\psi(\gamma_e)$, while vertices lie in $\mathcal{R}_\psi(\gamma_{L(v)}) \cap \mathcal{R}_\psi(\gamma_{R(v)})$.

Besides, since $Z(\gamma)$ is holomorphic, initial rays must originate from attractor points on the boundary.

Flow trees are subsets of scattering diagrams, determining sequences of scatterings which produce an outgoing ray $\mathcal{R}_{\psi}(\gamma)$ passing through the desired point $z$. 

B. Pioline (LPTHE, Paris)
Outline

1. Introduction
2. Kähler moduli space of $K_{\mathbb{P}^2}$
3. Attractor flow trees and scattering diagrams
4. Large volume scattering diagram
5. Towards the exact scattering diagram
For the large volume stability conditions $Z_{(s,t)}^{LV}$, [Bousseau’19] constructed the scattering diagram $\mathcal{D}_\psi$ in $(s, t)$ upper half-plane for $\psi = 0$. For $\psi \neq 0$, just map $(s, t) \mapsto (s - t \tan \psi, t / \cos \psi)$. 

Think of $R(\gamma)$ as the worldline of a fictitious particle of charge $r$, mass $m^2 = \frac{1}{2} d^2 - r \cosh^2$ moving in a constant electric field $E$. 

B. Pioline (LPTHE, Paris)
Large volume scattering diagram

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- The rays $\mathcal{R}(\gamma)$ are branches of hyperbola asymptoting to $t = \pm(s - \frac{d}{r})$ for $r \neq 0$, or vertical lines when $r = 0$. Walls of marginal stability $\mathcal{W}(\gamma, \gamma')$ are half-circles centered on real axis.
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The rays $\mathcal{R}(\gamma)$ are branches of hyperbola asymptoting to $t = \pm (s - \frac{a}{r})$ for $r \neq 0$, or vertical lines when $r = 0$. Walls of marginal stability $\mathcal{W}(\gamma, \gamma')$ are half-circles centered on real axis.

Think of $\mathcal{R}(\gamma)$ as the worldline of a fictitious particle of charge $r$, mass $m^2 = \frac{1}{2} d^2 - r \sinh^2$ moving in a constant electric field!
Initial rays correspond to $\mathcal{O}(m)$ and $\mathcal{O}(m)[1]$, i.e. (anti)D4-branes with $m$ units of flux, emanating from $(s, t) = (m, 0)$ on the boundary where the central charge vanishes.
Initial rays correspond to $O(m)$ and $O(m)[1]$, i.e., (anti)D4-branes with $m$ units of flux, emanating from $(s, t) = (m, 0)$ on the boundary where the central charge vanishes.

The first scatterings occur for $t \geq \frac{1}{2}$, after each constituent has moved by $|\Delta s| \geq \frac{1}{2}$. Causality and monotonicity of the ‘electric potential’ $\varphi(\gamma) = d - sr$ along the flow, allow to bound the number and charges of constituents.
Flow trees for $\gamma = [0, 4, 1)$

- $\{\{-3\mathcal{O}(-2), 2\mathcal{O}(-1)\}, \mathcal{O}\}$: 
  $3\mathcal{O}(-2) \rightarrow 2\mathcal{O}(-1) \oplus \mathcal{O} \rightarrow \mathcal{E}$ 
  $K_3(2, 3)K_{12}(1, 1) \rightarrow -156$

- $\{\mathcal{O}(-3), \{-\mathcal{O}(-1), 2\mathcal{O}\}\}$: 
  $\mathcal{O}(-3) \oplus \mathcal{O}(-1) \rightarrow 2\mathcal{O} \rightarrow \mathcal{E}$ 
  $K_3(1, 2)K_{12}(1, 1) \rightarrow -36$

Total: $\Omega_{\infty}(\gamma) = -192 = GV_4^{(0)}$
Flow trees for $\gamma = [1, 0, -3)$

- $\{\{-\mathcal{O}(-5), \mathcal{O}(-4)\}, \mathcal{O}(-1)\}$
  - $\mathcal{O}(-5) \to \mathcal{O}(-4) \oplus \mathcal{O}(-1) \to E$
  - $K_3(1, 1)^2 \to 9$

- $\{-\mathcal{O}(-4), \mathcal{O}(-3)\}$
  - $\mathcal{O}(-4) \oplus \mathcal{O}(-3) \to \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \to E$
  - $K_3(1, 1)^2 K_3(1, 2) \to 27$

- $\{-\mathcal{O}(-4), 2\mathcal{O}(-2)\}$
  - $\mathcal{O}(-4) \to 2\mathcal{O}(-2) \to E$
  - $K_6(1, 2) \to 15$

Total: $\Omega_\infty(\gamma) = 51 = \chi(\text{Hilb}_4\mathbb{P}^2)$
1 Introduction

2 Kähler moduli space of $K_{\mathbb{P}^2}$

3 Attractor flow trees and scattering diagrams

4 Large volume scattering diagram

5 Towards the exact scattering diagram
The full scattering diagram should interpolate between $\mathcal{D}^{LV}_\psi$ around $\tau = \infty$ and $\mathcal{D}^0_\psi$ around $\tau = \tau_0$, and be invariant under the action of $\Gamma_1(3)$. 
The full scattering diagram should interpolate between $\mathcal{D}^{LV}_\psi$ around $\tau = i\infty$ and $\mathcal{D}^o_\psi$ around $\tau = \tau_0$, and be invariant under the action of $\Gamma_1(3)$.

Under $\tau \mapsto \frac{\tau}{3n\tau + 1}$ with $n \in \mathbb{Z}$, $\mathcal{O} \mapsto \mathcal{O}[n]$. Hence we have an doubly infinite family of initial rays associated to $\mathcal{O}(m)[n]$. 
The full scattering diagram should interpolate between $D_{\psi}^{LV}$ around $\tau = i\infty$ and $D_{\psi}^{O}$ around $\tau = \tau_o$, and be invariant under the action of $\Gamma_1(3)$.

Under $\tau \mapsto \frac{\tau}{3n\pi + 1}$ with $n \in \mathbb{Z}$, $O \mapsto O[n]$. Hence we have an doubly infinite family of initial rays associated to $O(m)[n]$.

For $|\tan \psi| < \frac{1}{2V}$ where $V = \text{Im} T(0) = \frac{27}{4\pi^2} \text{ImLi}_2(e^{2\pi i/3}) \simeq 0.463$ only the rays associated to $O(m)[0]$ and $O(m)[1]$ escape to $i\infty$, and merge onto rays in the large volume scattering diagram $D_{\psi}^{LV}$. 
Exact scattering diagram - $\psi = 0$
In addition, there must be an infinite family of initial rays coming from $\tau = \frac{p}{q}$ with $q \neq 0 \mod 3$, corresponding to $\Gamma_1(3)$-images of $\mathcal{O}(0)$. 
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This includes initial rays emitted at \( \tau = n - \frac{1}{2} \), associated to \( \Omega(n+1) \); for \( \psi \sim \frac{\pi}{2} \), these merge onto initial rays of the orbifold scattering diagram.
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This includes initial rays emitted at \( \tau = n - \frac{1}{2} \), associated to \( \Omega(n + 1) \); for \( \psi \sim \frac{\pi}{2} \), these merge onto initial rays of the orbifold scattering diagram.

We conjecture that the only initial rays are the \( \Gamma_1(3) \) images of the structure sheaf \( \mathcal{O} \), each of them carrying \( \Omega(k\gamma) = 1 \) for \( k = 1, 0 \) otherwise.
For $\psi = \pm \frac{\pi}{2}$, the diagram $\mathcal{D}_\psi$ simplifies dramatically, since the loci $\text{Im} Z_\tau(\gamma) = 0$ are lines of constant $s := \frac{\text{Im} T_D}{\text{Im} T} = \frac{d}{r}$. 

![Exact scattering diagram](image-url)
For $\psi = \pm \frac{\pi}{2}$, the diagram $D_{\psi}^\Pi$ simplifies dramatically, since the loci $\text{Im}Z_{\tau}(\gamma) = 0$ are lines of constant $s := \frac{\text{Im}T_{D}}{\text{Im}T} = \frac{d}{r}$.

Hence, there is no wall-crossing between $\tau_0$ and $\tau = i\infty$ when $-1 \leq \frac{d}{r} \leq 0$, explaining why the Gieseker index $\Omega_{\infty}(\gamma)$ agrees with the index $\Omega_{c}(\gamma)$ in the anti-attractor chamber.
Exact scattering diagram, varying $\psi$

$\gamma = [0, 1, 1) = \text{ch } \mathcal{O}_C$:

$\gamma = [1, 0, 1) = \text{ch } \mathcal{O}$:
The scattering diagram is the proper mathematical framework for the attractor flow tree formula in the case of local CY3. This is because $Z(\gamma)$ is holomorphic on $M_K$, so the gradient flow preserves the phase of $Z(\gamma)$. Moreover, initial rays can only start from the boundary.
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This provides an effective way of computing (unframed) BPS invariants in any chamber, and a natural decomposition into elementary constituents. Mathematically, different trees should correspond to different strata in $\mathcal{M}_{Z(\gamma)}$.
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It would be interesting to extend this description to other toric CY3, such as local del Pezzo surfaces, and to framed BPS indices.
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For a compact CY3, $\arg Z(\gamma)$ is no longer constant along the flow and there can be attractor points with $\Omega_\star(\gamma) \neq 0$ at finite distance in Kähler moduli space.
Thanks for your attention!