Knizhnik-Zamolodchikov equations revisited:

(non)supersymmetric gauge theories, defects, and localization

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It is a common misconception, that exact computations in QFT
done using supersymmetric localization
only concern boring quantities like $n_B - n_F$,
the differences of the numbers of vacua

In fact, in 1992 Witten showed that the two dimensional Yang-Mills theory,

$$Z_\Sigma(g^2 Area_\Sigma) = \int DA\, e^{-\frac{1}{4g^2}\int_\Sigma \mathrm{tr} F_A \wedge \star F_A}$$

which, on the one hand, can be exactly solved using Migdal's method

$$Z_\Sigma(g^2 Area_\Sigma) = \sum_{\lambda \in Rep(G)} \mathrm{dim}(\lambda)^{2-2g_\Sigma}\, e^{-g^2 Area_\Sigma c_2(\lambda)}$$

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$$Z_\Sigma(g^2 Area_\Sigma) = \sum_{\lambda \in Rep(G)} \dim(\lambda)^{2-2g_\Sigma} e^{-g^2 Area_\Sigma c_2(\lambda)}$$

Admits another expression

$$Z_\Sigma(g^2 Area_\Sigma) =$$

$$\int_{\mathcal{M}_\Sigma^{\mathrm{flat}}(G)} e^{\omega + g^2 Area_\Sigma \Theta} +$$

$$\sum_{c:abelian\ YM\ connections} e^{-\frac{\|c\|^2}{g^2 Area_\Sigma}} \times computable\ \ prefactors$$

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which is a version of Duistermaat-Heckmann formula

$$\int_X e^{\omega+\langle\mu,\phi\rangle} = \sum_{p\in fixed\ \ points} \frac{e^{\langle\mu_p,\phi\rangle}}{\prod_i w_i(\phi)}$$

with

$$\mu : X \to \mathfrak{g}^*$$

a moment map of a group G action on a symplectic manifold (X, ω),

$$\int_X e^{\omega-\langle\mu,\mu\rangle} = \sum_{p\in fixed\ \ points} \text{"}erf-functions\text{"}\ associated\ to\ p \times e^{-\langle\mu_p,\mu_p\rangle}$$

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Witten's approach was to interpret the two dimensional YM

as a "sub-sector" of two dimensional supersymmetric Yang-Mills theory

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Witten's approach was to interpret the two dimensional YM

as a "sub-sector" of two dimensional supersymmetric Yang-Mills theory

So that the states of YM are the (bosonic) vacua
of the (suitably deformed) super-Yang-Mills

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In this way one can compute

$$\mathrm{Tr}\, e^{\beta\Delta}$$

for compact Lie groups, both representation-theoretically (Casimirs and reps) and geometrically (sums over geodesics = multiply wound loops on max torus)

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A closely related problem is that of the Calogero-Moser system

$$H_2 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \nu(\nu - 1) \sum_{i<j} U(x_i - x_j)$$

which, for

$$U(x) = \frac{1}{4\sin^2(x/2)}$$

can be found "inside" the two dimensional $U(N)$ Yang-Mills theory... for integer ν.

A. Gorsky, *NN*'93; *More recently N. Reshetikhin, in greater generality*

so one can use supersymmetric localization

to compute spectrum and wavefunctions in this non-supersymmetric problem

That was our starting question in 1992

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The supersymmetric localization applied to the appropriate gauge theory

Connects various domains of mathematical physics

Hyperbolic geometry, Bethe ansatz, Isomonodromic equations

Invariants of 3- and 4-manifolds

$\ldots$

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Non-integer ν not so innocent

Need to be sometimes compact, sometimes non-compact

Cf. quantum gravity/worldsheet string theory

E.Witten '2013

So that the ambient supersymmetric theory lives in more dimensions

Four dimensional $\mathcal{N} = 2$ theories

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Related circle of questions

In quantum field theory and statistical mechanics
one often uses the trick of analytic continuation from $\mathbb{Z}$ to $\mathbb{C}$

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Particles as S-matrix poles in complex angular momentum l

T.Regge

Replica trick: $\langle \log Z \rangle_J \to \langle Z^n \rangle_J$

G.Parisi

Dimensional regularization: spacetime dimension D

G.'tHooft

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Is there any physical meaning to these complexifications?

What physical system realizes complex spin representations of $\mathfrak{sl}_N$?

Which physical system's partition function is equal to Z^n for complex n?

In string paradigm the number of species is the spacetime dimension

$D \sim c$, the central charge of the matter sector of the worldsheet theory

What is the physical realization of Virasoro representations with complex c?

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Is there any physical meaning to these complexifications?

We shall argue the answer is in extra dimensions and supersymmetry!

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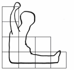
Our first story: Generalization of Dyson-Macdonald identities

$$\eta(\mathfrak{q})^{-\dim(G)} = \sum_{\lambda} \tau_{\lambda} \mathfrak{q}^{|\lambda|}$$

to



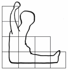
Picture of arms and legs by Ugo Bruzzo

$$\eta(\mathfrak{q})^{\frac{(m+\varepsilon_1)(m+\varepsilon_2)}{\varepsilon_1 \varepsilon_2}} =$$

$$= \sum_{\lambda} \prod_{\square \in \lambda} \frac{(m+\varepsilon_1(a_\square+1)-\varepsilon_2 l_\square)(m-\varepsilon_1 a_\square+\varepsilon_2(l_\square+1))}{(\varepsilon_1(a_\square+1)-\varepsilon_2 l_\square)(-\varepsilon_1 a_\square+\varepsilon_2(l_\square+1))} \mathfrak{q}^{|\lambda|}$$

Generalization of Dyson-Macdonald identities

to



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SUPERSYMMETRIES AND REPLICAS

is there a physical realization of the replica trick? could one refine it?

since the replica symmetry is often broken,

could one introduce some chemical potentials for different S(n) representations?

SUPERSYMMETRIES AND REPLICAS

$\lambda \in \mathbb{C}$

$$\frac{1}{\eta(q)^{\lambda}} = \sum_{\rho} q^{|\rho|} \times \mu_{\rho}(\lambda)$$

For $\lambda \in \mathbb{Z}$ $\quad |\lambda|$ chiral bosons/fermions

flavor

$$\lambda = \frac{(m+\varepsilon_1)(m+\varepsilon_2)}{\varepsilon_1 \varepsilon_2}$$

spacetime rotations

Partition function of 4d gauge theory $\mathcal{N}=2$ / can be refined to 6d tensor theory $(2,0)$

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For theories with O(n) or U(n) symmetry one can use Deligne category to define "representations" for complex n… (Binder-Rychkov'2016)

For Chern-Simons theory with a simple Lie gauge group one can use Vogel plane to define universal CS theory (Mkrtchyan-Veselov'2012)

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Partition function of 4d gauge theory $\mathcal{N}=2$ / can be refined to 6d tensor theory (2,0)

The refined replica of 2d chiral bosons/fermions = 6d (2,0) tensor multiplet

Q: Refined replica of 2d chiral WZW ADE theory = nonabelian 6d (2,0) SCFT theory?

SUPERSYMMETRIES AND REPLICAS

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The refined replica of 2d chiral bosons/fermions = 6d (2,0) tensor multiplet

Q: Refined replica of 2d chiral WZW ADE theory = nonabelian 6d (2,0) SCFT theory?

The refined replica of 3d conformally coupled scalar = 11d linearized supergravity

(NN conjecture 2004, A.Okounkov proof 2015)

Q: what is the ``non-abelian'' 3d theory whose *replicant is* M-theory?

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Our next example: complexification of Chern-Simons theory

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Let us start with the simple representation theory

of $\mathfrak{sl}_2$ algebra

$$L_+ = x^2\partial_x - 2sx\,,\;\; L_0 = x\partial_x - s\,,\;\; L_- = \partial_x$$

Realized in $\psi(x)dx^{-s}$ tensors in one dimension.

For $2s \in \mathbb{Z}_+$ there is a finite dimensional $SL(2,\mathbb{C})$ group representation

$$\psi(x) = f_0 + f_1 x + \ldots + f_{2s}x^{2s}$$

$$\psi(x)dx^{-s} \mapsto f\left(\frac{ax+b}{cx+d}\right)(cx+d)^{2s}dx^{-s}$$

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For $2s \in \mathbb{Z}_+$ there is a finite dimensional $SL(2,\mathbb{C})$ group representation

$$\psi(x) = \psi_0 + \psi_1 x + \ldots + \psi_{2s} x^{2s}$$

The space of states of a quantum mechanics of a particle on a sphere S^2

$_{Geometric\ \ quantization, Kirillov-Kostant-Souriau}$

$$\int DpDq\, e^{\mathrm{i}\int p\dot{q}}$$

$$dp \wedge dq = \mathrm{i}s \frac{dx \wedge d\bar{x}}{(1 + x\bar{x})^2}$$

The symmetry of quantum mechanics is $SU(2)$

The wavefunction $\psi(x)$ is a globally defined holomorphic section of $\mathcal{O}(2s)$

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Once $s \in \mathbb{C}$ the group action is lost

There are various options for the nature of the $\psi(x)$ functions

Verma modules $\mathcal{V}_s^+$: $\psi(x) =$ a polynomial in x

Verma modules $\mathcal{V}_s^-$: $\psi(x) = x^{2s} \cdot$ a polynomial in x^{-1}

Heisenberg-Weyl modules $\mathcal{HW}_s^a$: $\psi(x) = x^{s+a} \cdot$ a polynomial in x, x^{-1}

No hermitian invariant product for generic $s_1, s_2, s_3 \in \mathbb{C}$

Only the Lie algebra $\mathfrak{sl}_2$ acts

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We encounter these representations when we think about invariants

$$\mathcal{J}^{s_1,s_2,s_3} = (x_1 - x_2)^{s_1+s_2-s_3}(x_2 - x_3)^{s_2+s_3-s_1}(x_1 - x_3)^{s_1+s_3-s_2}$$

Is invariant under $L_n^{(1)} + L_n^{(2)} + L_n^{(3)}$
Expand $\mathcal{J}^{s_1,s_2,s_3}$ in the region

$$|x_1| \ll |x_2| \ll |x_3|$$

to see $\mathcal{J}^{s_1,s_2,s_3} \in \mathcal{V}^+_{s_1} \otimes \mathcal{HW}^{s_1-s_3}_{s_2} \otimes \mathcal{V}^-_{s_3}$

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The next stop is the Knizhnik-Zamolodchikov equation

$$\boldsymbol{\Psi} = \mathcal{J}^{s_0, s_1, \ldots, s_{n+1}} \in \left(\mathcal{V}^{+}_{s_0} \otimes \mathcal{HW}^{a_1}_{s_1} \otimes \mathcal{HW}^{a_2}_{s_2} \otimes \ldots \otimes \mathcal{V}^{-}_{s_{n+1}} \right)^{\mathfrak{sl}_2}$$

depending on additional parameters $z_0, z_1, \ldots, z_{n+1} \in \mathbb{CP}^1$
obeying a system of compatible(!) equations

$$\nabla_i \boldsymbol{\Psi} \equiv (k+2) \frac{\partial}{\partial z_i} \boldsymbol{\Psi} - \widehat{H}_i \boldsymbol{\Psi} = 0$$

with z-dependent Gaudin Hamiltonians

$$\widehat{H}_i = -\sum_{j \neq i} \frac{1}{z_i - z_j} \left(x_{ij}^2 \frac{\partial^2}{\partial x_i \partial x_j} - 2x_{ij} \left(s_i \frac{\partial}{\partial x_j} - s_j \frac{\partial}{\partial x_i} \right) - 2 s_i s_j \right)$$

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For $2s_i \in \mathbb{Z}_+$ one can restrict $\boldsymbol{\Psi}$ to be polynomials in x_i of degree $2s_i$

For $k \in \mathbb{Z}_+$ finite dimensional space of solutions

conformal blocks of $SU(2)_k$ Wess-Zumino-Novikov-Witten theory

Tsuchiya—Ueno—Yamada

$$(k+2)\frac{\partial}{\partial z_i}\boldsymbol{\Psi} - \widehat{H}_i\boldsymbol{\Psi} = 0$$

with z-dependent Gaudin Hamiltonians

$$\widehat{H}_i = \sum_{j\neq i}\frac{1}{z_i - z_j}\left(L_+^{(i)}L_-^{(j)} + L_+^{(j)}L_-^{(i)} - 2L_0^{(i)}L_0^{(j)}\right)$$

Interpreted in $3d$ Chern-Simons theory
as equations for the parallel transport of a quantum state
in geometric quantization of the moduli space
of flat $SU(2)$ connections over n-punctured sphere

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Mathematicians and physicists have studied these equations for generic $k \in \mathbb{C}$

Feigin–Frenkel, Reshetikhin, Babujian–Flume, Feigin–Varchenko–Schekhtman...

conformal blocks of level k $\widehat{\mathfrak{sl}_2}$ current algebra

What is the physics? For complex s_i's and k's?

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The answer is provided by the four dimensional gauge theory

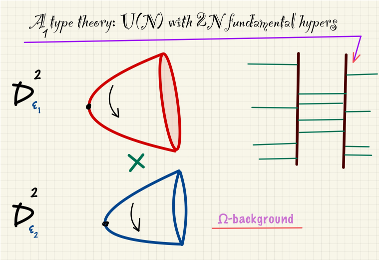
Our main example today will be
the super-Yang-Mills with fundamental matter
subject to Ω-deformation

$$ds^2 = ds^2_{\mathcal{D}_1^2} + ds^2_{\mathcal{D}_2^2}$$

$$ds^2_{\mathcal{D}_i^2} = f_i(r_i)\left(dr_i^2 + r_i^2 d\varphi_i^2\right)$$

$$i = 1, 2$$

$$V = \varepsilon_1 \partial_{\varphi_1} + \varepsilon_2 \partial_{\varphi_2}$$
$$\bar{V} = \bar{\varepsilon}_1 \partial_{\varphi_1} + \bar{\varepsilon}_2 \partial_{\varphi_2}$$



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First of all, we can compute exactly quite a few things

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We can compute its super-partition function

$$\mathcal{Z}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}\right)$$

$$= \int_{gauge\ \ fields\ +\ matter +\ superpartners} DAD\psi D\sigma D\bar{\sigma} D\chi D\eta\ e^{-\mathcal{S}_{\varepsilon_1, \varepsilon_2}}$$

where we fix the asymptotics $\sigma(x) \to \mathrm{diag}\left(a_1, \ldots, a_N\right)$ as $x \to \infty$

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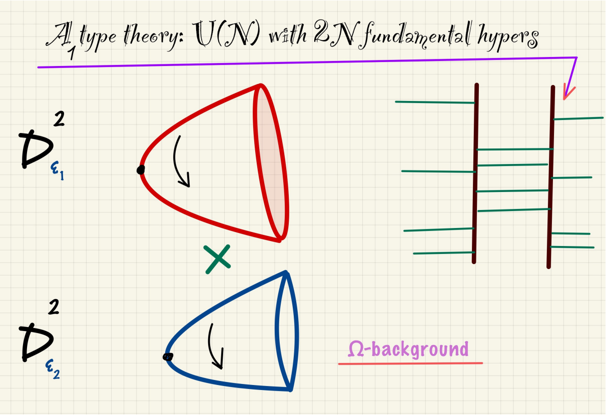
We can compute its super-partition function using localization and other clever tricks

$$\mathcal{Z}\,(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q})$$



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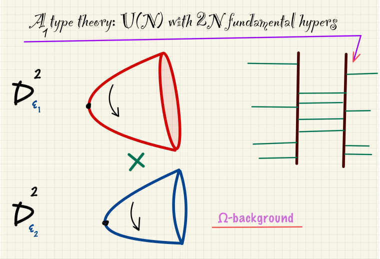


$$\mathbf{a} = (a_1, \ldots, a_N)\ ,\ \mathbf{m} = \left(m_1^\pm, \ldots, m_N^\pm\right)\ ,\ \mathfrak{q} = e^{2\pi i\tau}\ ,\ \tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g_{\mathrm{ym}}^2}$$

We can compute its super-partition function
using localization and other clever tricks

$$\mathcal{Z}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}) = \mathcal{Z}^{\mathrm{pert}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}) \mathcal{Z}^{\mathrm{inst}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q})$$

$$\mathcal{Z}^{\mathrm{inst}}(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathfrak{q}) = \sum_{\lambda^{(1)},\ldots,\lambda^{(N)}} \prod_{\alpha=1}^{N} \mathfrak{q}^{|\lambda^{(\alpha)}|} \times$$

$$\times \prod_{\alpha,\beta=1}^{N} \frac{\prod_{(i,j)\in\lambda^{(\alpha)}} \left(a_\alpha - m_\beta^{+} + c_{i,j}\right)\left(m_\beta^{-} - a_\alpha - c_{i,j}\right)}{\prod_{(i,j)\in\lambda^{(\alpha)}} \prod_{(i',j')\in\lambda^{(\beta)}} \left(a_\alpha - a_\beta + d_{i,j;i',j'}\right)}$$

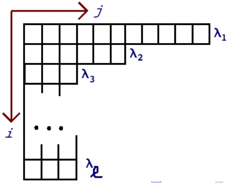
$\mathbf{a} = (a_1,\ldots,a_N)$ Coulomb moduli

$\mathbf{m} = (m_1^{\pm},\ldots,m_N^{\pm})$ Masses of fundamental hypers

$\mathfrak{q} = e^{2\pi i\tau}$ Instanton fugacity

$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g_{\mathrm{ym}}^2}$ Complexified gauge coupling

$$c_{i,j} = \varepsilon_1(i-1) + \varepsilon_2(j-1)$$



In the *classical* limit $\varepsilon_1, \varepsilon_2 \to 0$

$$\mathcal{Z}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}\right) \;=\; \exp \frac{1}{\varepsilon_1 \varepsilon_2} \mathcal{F}\left(\mathbf{a}, \mathbf{m}; \mathfrak{q}\right)$$

with the special geometry of an algebraic integrable system emerging

genus zero $SL(N)$ Hitchin system $=$ classical Gaudin

Prepotential $\mathcal{F}(\mathbf{a}, \mathbf{m}; \mathfrak{q})$ of classical Gaudin:

Spectral curve $\mathcal{C}_{\mathbf{u}}$: $\mathrm{Det}\left(\sum_I \frac{\Phi_I}{\xi-\xi_I} - \eta \cdot \mathbf{1}_N\right) = 0$

$$\Phi_0 + \Phi_{\mathfrak{q}} + \Phi_1 + \Phi_\infty = 0$$

$$\begin{aligned}
\Phi_0 &\sim \mathrm{diag}\left(m_1^+ - m^+, \ldots, m_N^+ - m^+\right), \\
\Phi_{\mathfrak{q}} &\sim \mathrm{diag}\left(m^+, \ldots, m^+, m^+(1-N)\right) \\
\Phi_1 &\sim \mathrm{diag}\left(m^-, \ldots, m^-, m^-(1-N)\right), \\
\Phi_\infty &\sim \mathrm{diag}\left(m_1^- - m^-, \ldots, m_N^- - m^-\right)
\end{aligned}$$

$$\begin{aligned}
Nm^+ &= m_1^+ + \ldots + m_N^+, \\
Nm^- &= m_1^- + \ldots + m_N^-
\end{aligned}$$

$$a_i = \oint_{A_i} \eta d\xi \,, \qquad \frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B^i} \eta d\xi$$

Hamiltonians H_i, $i = 1, \ldots, N-1$ of classical Gaudin:

$$\mathrm{Det}\left(\sum_I \frac{\Phi_I}{\xi - \xi_I} - \eta \cdot \mathbf{1}_N\right) = \sum_{I,l} \frac{\mathcal{J}_{I,l}\,\eta^{N-l}}{(\xi - \xi_I)^l}$$

Quite a few relations
$N-1$ independent ones:

$$H_i = \mathrm{tr}\left(\Phi_0 + \Phi_{\mathfrak{q}}\right)^i\ ,\ \ i = 2, \ldots, N$$

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That was a classical integrable system

In the limit $\varepsilon_1, \varepsilon_2 \to 0$ of four dimensional gauge theory

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That was a classical integrable system

Now turn the Ω-deformation back: $\varepsilon_1, \varepsilon_2 \neq 0$:

Quantum version of isomonodromic deformation!

N. Reshetikhin'91

Knizhnik-Zamolodchikov/quantum differential equation

Two dimensional version of instanton partition function *Givental'94*

$$\kappa \frac{\partial \boldsymbol{\Psi}}{\partial z_i} = \widehat{H}_i \cdot \boldsymbol{\Psi}$$

Two quasiclassical limits

- $\kappa \to 0$, $\boldsymbol{\Psi} = e^{\frac{\tilde{W}}{\kappa}} \cdot \chi$

$$\widehat{H}_i \chi = E_i \chi\,,\ E_i = \frac{\partial \tilde{W}}{\partial z_i}$$

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Quantum version of isomonodromic deformation

Knizhnik-Zamolodchikov/quantum differential equation

$$\kappa \frac{\partial \mathbf{\Psi}}{\partial t^i} = \widehat{H}_i \cdot \mathbf{\Psi}$$

Two quasiclassical limits

- $\kappa \to \infty$, $\mathbf{\Psi} = e^{\kappa S} \cdot \tilde{\chi}$

$$\frac{\partial S}{\partial z_i} = H_i \left(\frac{\partial S}{\partial \mathbf{x}}, \mathbf{x}; \mathbf{z} \right)$$

up to little symplectic subtleties of keeping something fixed

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FOUR DIMENSIONAL TOYS

Surface defects

Kronheimer+Mrowka'93–95

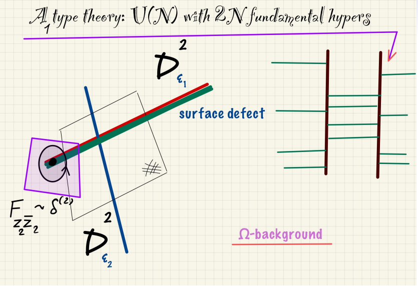
Losev+Moore+NN+Shatashvili'95

NN'95,NN'04

Braverman'04

Gukov+Witten'08

Kanno+Tachikawa'11



$$\boldsymbol{\Psi}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}\right) = \boldsymbol{\Psi}^{\rm pert}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}\right)\boldsymbol{\Psi}^{\rm inst}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}\right)$$

$$= \mathfrak{q}^{\frac{\mathbf{a}^2}{2\varepsilon_1\varepsilon_2}} \prod_{\omega} w_{\omega}^{\frac{a_\omega - a_{\omega+1}}{\varepsilon_1}} \times \sum_{\lambda^{(1)},\ldots,\lambda^{(N)}} \prod_{\omega} w_{\omega}^{k_\omega(\lambda)} \mathfrak{q}^{k_{\rm bulk}(\lambda)} \times$$

$$\times \left(\prod_{\alpha,\beta=1}^{N} \frac{\prod_{(i,j)\in\lambda^{(\alpha)}} \left(a_\alpha - m_\beta^{+} + c_{i,j}\right)\left(m_\beta^{-} - a_\alpha - c_{i,j}\right)}{\prod_{(i,j)\in\lambda^{(\alpha)}} \prod_{(i',j')\in\lambda^{(\beta)}} \left(a_\alpha - a_\beta + d_{i,j;i',j'}\right)} \right)^{\mathbb{Z}_N}$$

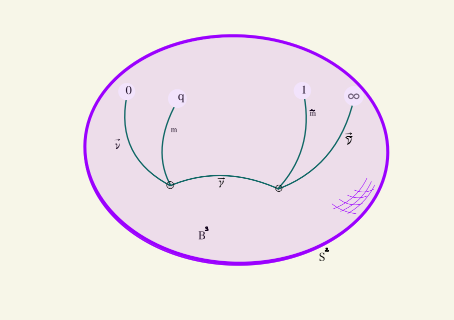
NN′04

Regular surface defect partition function

$$\boldsymbol{\Psi}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q}\right) =$$

Solves 4-point Knizhnik-Zamolodchikov equation

Theorem by NN+Tsymbalyuk′17−21



BPS/CFT correspondence

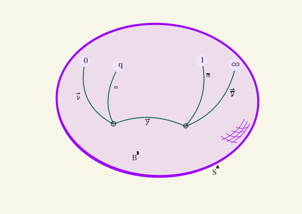
NN′04

Regular surface defect partition function

$$\boldsymbol{\Psi}\left(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}\right)\ =$$

Solves 4-point Knizhnik-Zamolodchikov equation
with $\boldsymbol{\Psi} \in \left(\mathcal{V}^{+} \otimes \mathcal{HW} \otimes \mathcal{HW} \otimes \mathcal{V}^{-}\right)^{\mathfrak{sl}_N}$

Theorem by NN+Tsymbalyuk′17−21



For $n=4$, $N=2$ it is PVI

Corollary:

$\varepsilon_1 \to 0$ *isomonodromic τ−function*

$$\boldsymbol{\Psi} \sim e^{\frac{\log \tau}{\varepsilon_1}}$$

Using BPZ equations observed earlier by

Teschner′15

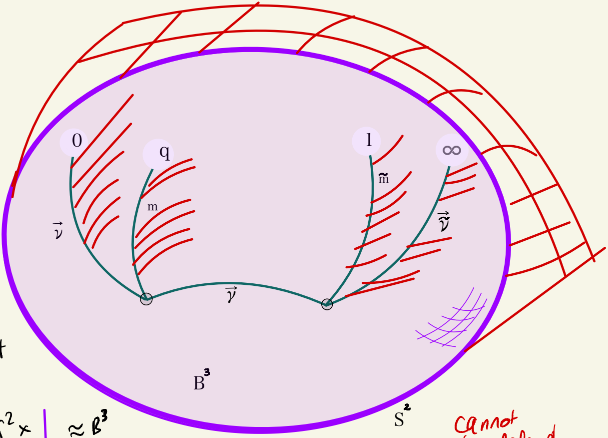
Litvinov+Lukyanov+NN+Zamolodchikov′16

Regular surface defect in $\mathcal{N} = 2$ vs surface junction in $\mathcal{N} = 4$

$$\boldsymbol{\Psi}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q}\right) \in \left(\mathcal{V}^+ \otimes \mathcal{HW} \otimes \mathcal{HW} \otimes \mathcal{V}^-\right)^{\mathfrak{sl}_N}$$

Solves 4-point Knizhnik-Zamolodchikov equation

Surface defects in $\mathcal{N}=4$ theory on $S^2 \times$



up to Q-exact terms $\approx$ CS

$S^2 \times | \approx B^3$

but electric charges in ∞-dim reps $\mathcal{V}^{\pm}, \mathcal{HW} \ldots$

cannot be defined as Wilson lines! surface ops!

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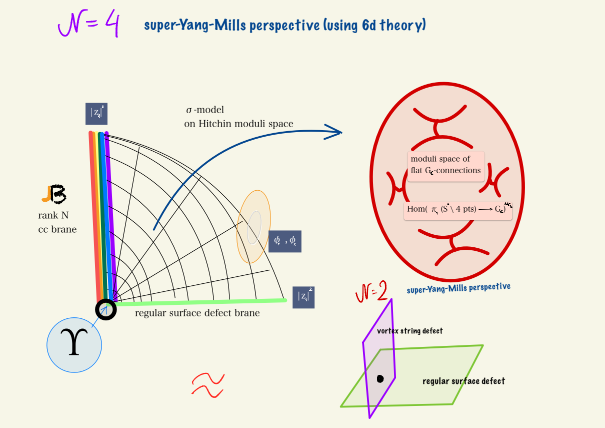
Intersecting regular and folded surface defect partition function

$$\widehat{\Psi}\,(\mathbf{a},\mathbf{m},\varepsilon_1,\varepsilon_2;\mathbf{w},\mathfrak{q}) \in \mathbb{C}^N$$

Solves 5-point Knizhnik-Zamolodchikov equation



Mixed complex spins and finite dimensional reps

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Parallel regular and folded surface defect partition function

$\tilde{\Psi}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q}\right)$

Solves 5-point Knizhnik-Zamolodchikov equation
with novel type of vertex operators: non-critical Hecke modifications

In progress by Jeong+Lee+NN'22

On $4d$ gauge theory side the observable is defined by

$$\langle S_R F(\mathbf{y}) \rangle = \sum_{(x_0, x_1, \dots, x_{N-1}) \in \mathbf{L}} \prod_{\omega=0}^{N-1} y_\omega^{\frac{x_\omega}{\varepsilon_1}} \left\langle \prod_{\omega=0}^{N-1} \mathcal{Q}_\omega(x_\omega) \right\rangle^{\mathbb{Z}_N}$$

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Folded surface defect partition function

In progress by Jeong+Lee+NN′22

On $4d$ gauge theory side the observable is defined by

$$\langle S_R F(\mathbf{y}) \rangle = \sum_{(x_0, x_1, \dots, x_{N-1}) \in \mathbf{L}} \prod_{\omega=0}^{N-1} y_\omega^{\frac{x_\omega}{\varepsilon_1}} \left\langle \prod_{\omega=0}^{N-1} \mathcal{Q}_\omega(x_\omega) \right\rangle^{\mathbb{Z}_N}$$

$\mathcal{Q}_\omega(x)$ - virtual Chern polynomial (actually, a function) of an infinite-dimensional bundle of Dirac zeromodes of the ω-component of the restriction of the gauge bundle onto the surface of the regular surface defect

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Parallel regular and folded surface defect in the limit $\varepsilon_2 \to 0$:

$$\tilde{\Psi} \sim e^{\frac{\tilde{W}(\mathbf{a},\mathbf{m},\varepsilon_1;\mathfrak{q})}{\varepsilon_2}} \chi\left(\mathbf{a},\mathbf{m},\varepsilon_1;\mathbf{w},\mathfrak{q}\right)$$

Where χ solves a 4-point oper

Hecke modifications, studied by Beilinson-Drinfeld

Geometric Langlands

Recently, for $N = 2$,

analytic Langlands correspondence of Etingof−Frenkel−Kazhdan

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General N case, away from the critical level $\varepsilon_2 \neq 0$, $k \neq -N$

Let us look at the $n+1$-point KZ equation

with the punctures at $z_0, z_1, \ldots, z_n$
with the special HW module attached to z_0

$$\mathbf{\Psi}(X^1, \ldots, X^N) = (X^N)^k \varphi\left(x^1, \ldots, x^{N-1}\right)$$

$$x^i = \frac{X^i}{X^N}, \quad i = 1, \ldots, N-1$$

$$\mathbf{J}^A_{0,B} = -X^A \frac{\partial}{\partial X^B} + \frac{k}{N} \delta^A_B$$

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General N case, away from the critical level $\varepsilon_2 \neq 0$, $k \neq -N$

Let us look at the $n+1$-point KZ equation

with the punctures at $z_0, z_1, \ldots, z_n$
with the special HW module attached to z_0

$$\mathfrak{D}_0 = (k+N)\frac{\partial}{\partial z_0} - \sum_{a=1}^{n} \frac{\mathbf{J}^A_{0,B}\mathbf{J}^B_{a,A}}{z_0 - z_a}$$

$$= (k+N)\nabla_0 + \sum_{i=1}^{N-1} \frac{\partial}{\partial x^i} \circ \mathbf{V}^i$$

where the operators $\nabla_0, \mathbf{V}^i$ mutually commute: $[\nabla_0, \mathbf{V}^i] = 0 = [\mathbf{V}^i, \mathbf{V}^j]$
and form the first class constraints with the rest of KZ equation

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General N case, away from the critical level $\varepsilon_2 \neq 0,\ k \neq -N$

$$\mathfrak{D}_0 = (k+N)\frac{\partial}{\partial z_0} - \sum_{a=1}^{n} \frac{\mathbf{J}^A_{0,B}\mathbf{J}^B_{a,A}}{z_0 - z_a}$$

$$= (k+N)\nabla_0 + \sum_{i=1}^{N-1} \frac{\partial}{\partial x^i} \circ \mathbf{V}^i$$

where the operators $\nabla_0, \mathbf{V}^i$ mutually commute: $[\nabla_0, \mathbf{V}^i] = 0 = [\mathbf{V}^i, \mathbf{V}^j]$ and form the first class constraints with the rest of KZ equation

$$[\nabla_0, \mathfrak{D}_a] = \sum_{i=1}^{N-1} \frac{{}^{N}_{a,i}\mathbf{V}^i}{z_0 - z_a}, \qquad a = 1, \ldots, n$$

$$[\mathfrak{D}_a, \mathbf{V}^i] = \frac{\mathbf{L}^N_s \mathbf{V}^i - {}^{i}_{a,j}\mathbf{V}^j + x^i \mathbf{T}^N_{a,j}\mathbf{V}^j}{z_0 - z_a}.$$

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General N case, away from the critical level $\varepsilon_2 \neq 0$, $k \neq -N$

$$\mathfrak{D}_0 = (k+N)\frac{\partial}{\partial z_0} - \sum_{a=1}^{n} \frac{\mathbf{J}_{0,B}^{A}\mathbf{J}_{a,A}^{B}}{z_0 - z_a}$$

$$= (k+N)\nabla_0 + \sum_{i=1}^{N-1} \frac{\partial}{\partial x^i} \circ \mathbf{V}^i$$

Thus, one can make a consistent truncation (reduction)

$$\nabla_0 \mathbf{\Psi} = 0\,,\ \mathbf{V}^i \mathbf{\Psi} = 0\,,\ \mathfrak{D}_a \mathbf{\Psi} = 0$$

The equations $\nabla_0 \mathbf{\Psi} = \mathbf{V}^i \mathbf{\Psi} = 0$ are first order PDEs in $z_0, x^1, \ldots, x^{N-1}$

Therefore: $\mathbf{\Psi}(z_0, z_1, \ldots, z_n; \vec{x}) \in V_1 \otimes \ldots \otimes V_n$

can be expressed, via a linear (Hecke) transformation,

through the n-point conformal block

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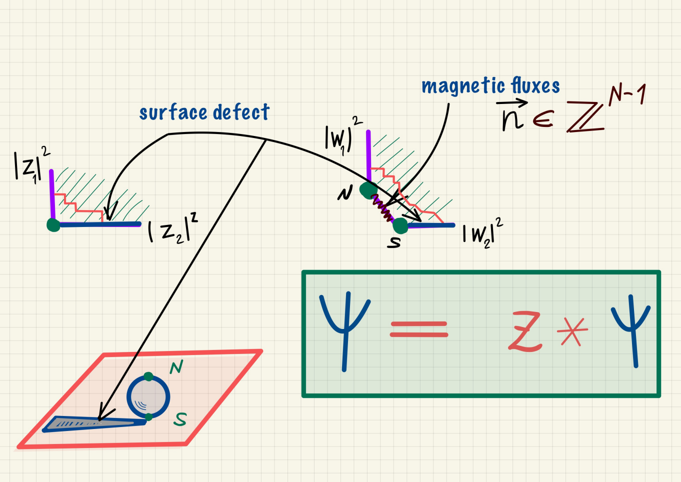
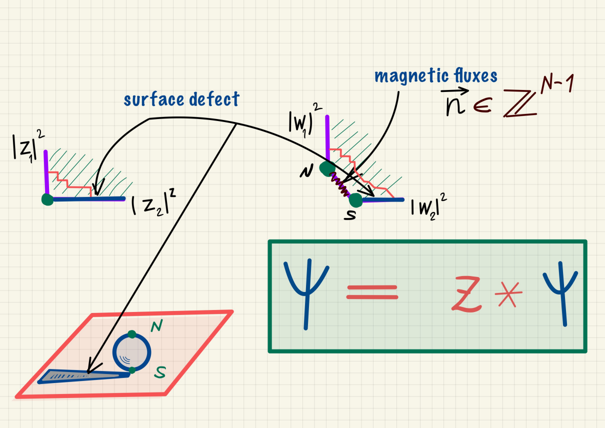
The power of four dimensions: Blown up Surface defects



The power of four dimensions: Blown up Surface defects



$$\boldsymbol{\Psi}\left(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q}\right) =$$

$$\sum_{\mathbf{n} \in \mathbb{Z}^{N-1}} \mathcal{Z}\left(\mathbf{a} + \varepsilon_2 \mathbf{n}, \mathbf{m}, \varepsilon_1 - \varepsilon_2, \varepsilon_2; \mathfrak{q}\right) \boldsymbol{\Psi}\left(\mathbf{a} + \varepsilon_1 \mathbf{n}, \mathbf{m}, \varepsilon_1, \varepsilon_2 - \varepsilon_1; \mathbf{w}, \mathfrak{q}\right)$$

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Limit $\varepsilon_1 \to 0$: higher rank analogue of GIL "Kyiv" formula

$N = 2, n = 4$ *case*: *Gamayun−Iorgov−Lysovyy*′12

$$\textit{Schematically}, \ \tau^{\mathrm{PVI}}_{\vec{\nu}}(a, b; \mathfrak{q}) = \sum_{n \in \mathbb{Z}} e^{nb} \mathcal{Z}_{\vec{\nu}}(a + n; \mathfrak{q})^{c=1}$$

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BPS/CFT correspondence

The $\mathbb{R}^4 \to \mathbb{R}^2$ reduction $\varepsilon_1 \to 0$

corresponds to $c \to \infty$, i.e. classical conformal blocks

$A.\ and\ \ Al.\ Zamolodchikov,\ \ late\ \ eighties$

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Going back to the Calogero-Moser system

The surface defect $\mathbf{\Psi}$ for the $\mathcal{N} = 2*$ theory

solves

$$\varepsilon_2 \varepsilon_1 \frac{d}{d\tau} \mathbf{\Psi} = \left[\sum_{i=1}^{N} \frac{\varepsilon_1^2}{2} \frac{\partial^2}{\partial x_i^2} + \varepsilon_3(\varepsilon_3 + \varepsilon_1) \sum_{i<j} \wp(x_i - x_j; \tau) \right] \mathbf{\Psi}$$

Taking $\varepsilon_2 \to 0$ limit is still complicated

some progress with N. Lee

Taking $\tau \to \mathrm{i}\infty$ limit is simple: get wavefunctions for hyperbolic CM

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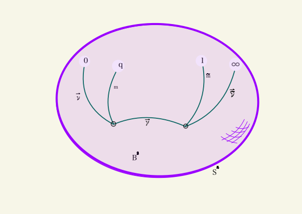
Another limit, $\varepsilon_3 \to \infty$, $\tau \to \mathrm{i}\infty$, $\Lambda = \varepsilon_3 e^{\frac{2\pi \mathrm{i}\tau}{N}}$ finite
Periodic Toda system

Can also be obtained from the Gaudin model we discussed above



Taking $\varepsilon_2 \to 0$ limit is doable
gives Sklyanin-like quantum separation of variables
generalizing integral formulas of Kharchev-Lebedev

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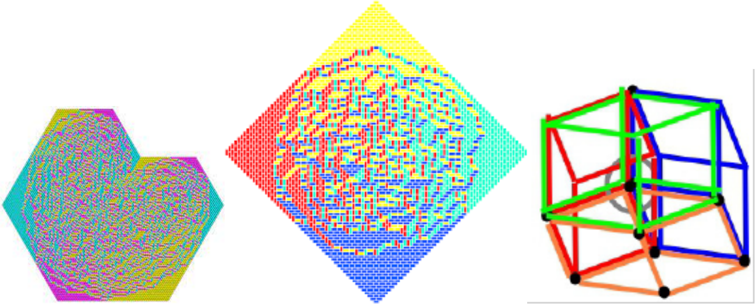
Future directions:



Connecting quantum geometry and integrability to (topological) string theory

Supersymmetric interfaces to generalize stable envelopes of Okounkov et al.

In progress with M.Dedushenko

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