Automorphic Spectra and the Conformal Bootstrap

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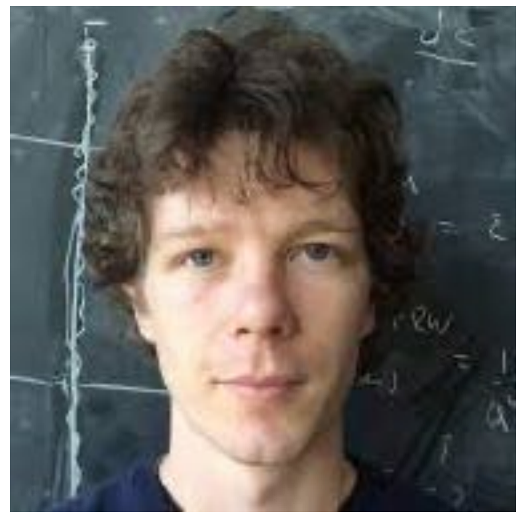
Based on **arXiv:2111.12716** with **Petr Kravchuk** and **Sridip Pal**.

Similar results appeared in **arXiv:2111.13215** by **James Bonifacio**.



I will also report on some ongoing work with all of the above.

What Is Quantum Field Theory?

1. How do we rigorously define quantum field theory?

2. How do we compute observables?

Situation better for **conformal field theories**:

- Precise axiomatic formulation in any number of dimensions.
- Effective for computations, even leading to new predictions.

} conformal bootstrap

CFT Axioms

1. V = a unitary representation of the conformal group in d dimensions.

   - V = space of states = space of local operators.
   - Decompose into irreducible representations: $V = \bigoplus_i V_{\Delta_i,\rho_i}$.
   - Local operators: $\mathcal{O}_i(x)$ with $x \in B^d$ generates V_{Δ_i,ρ_i}.

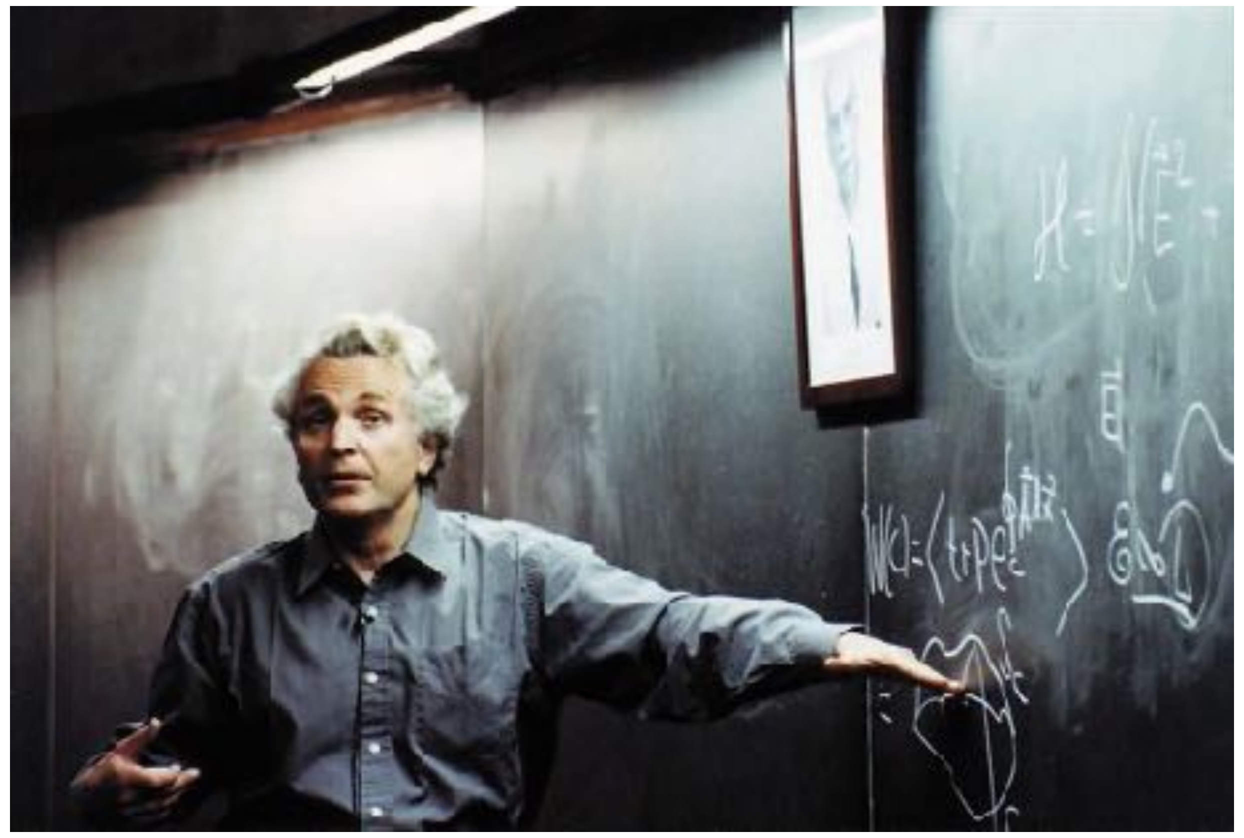
2. Operator product expansion: $\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k c_{ijk} |x - y|^{-\Delta_i - \Delta_j + \Delta_k} \mathcal{O}_k(y)$.

3. Associativity: $\mathcal{O}_i(x)(\mathcal{O}_j(y)\mathcal{O}_k(z)) = (\mathcal{O}_i(x)\mathcal{O}_j(y))\mathcal{O}_k(z)$

$\Rightarrow$ stringent constraints on the spectrum Δ_i, ρ_i and structure constants c_{ijk} .

Long term goal: Solve and classify CFTs in general dimension starting from these axioms.

A. Polyakov: "I was dreaming in the 1970s to have a classification of fixed points based on the operator product expansion. The program was successful in two dimensions, and I think it is not excluded that in three dimensions something like that is still possible."

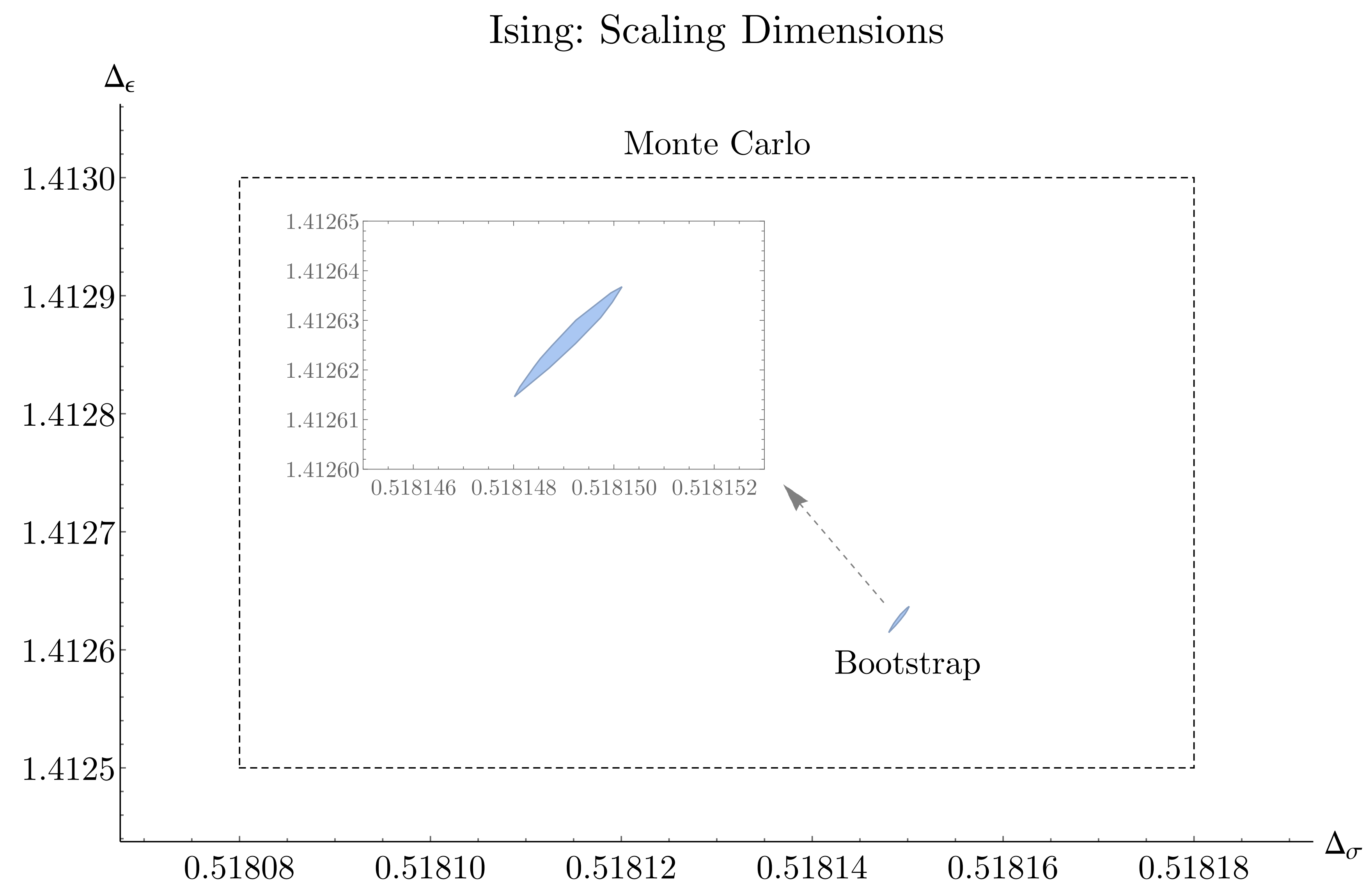


Current status:

- $d = 2$: partial progress (rational theories, Liouville theory), see talk by Vincent Vargas.
- $d > 2$: The only solved examples are free theories, but infinitely many interacting examples surely exist.

The CFT axioms seem capable of isolating interacting CFTs in $d > 2$: nearly sharp bounds on the spectral data from **linear and semidefinite programming**.

Circumstantial evidence in favor of Polyakov's dream.



[Kos, Poland Simmons-Dufin, Vichi '16]

Speculation: Can we solve an interacting CFT in d>2, such as the 3d Ising model?

Possible strategy: Identify a mathematical structure which produces spectral data satisfying the CFT axioms.

[Moore, Seiberg '89] [Gadde '17] [Guillarmou, Kupiainen, Rhodes, Vargas '21]

Today: Hyperbolic manifolds provide an excellent toy model for such structure.

Hyperbolic Manifolds

Definition: A hyperbolic d-manifold is a Riemannian d-manifold of constant sectional curvature -1.

The simplest example: Hyperbolic space $\mathbb{H}^d$.

- $-x_0^2 + x_1^2 + \ldots + x_d^2 = -1, x_0 > 0$
  $ds^2 = -dx_0^2 + dx_1^2 + \ldots + dx_d^2$
- Isometry group of $\mathbb{H}^d$ = $\mathrm{SO}^+(1,d)$

Fact: Every (closed, connected, orientable) hyperbolic d-manifold is of the form $\Gamma \backslash \mathbb{H}^d$, where Γ is a discrete subgroup of $\mathrm{SO}^+(1,d)$.

The Analogy

conformal field theories in d-1 dimensions ⟷ **hyperbolic d-manifolds**

Underlying reason:

conformal group of Euclidean $\mathbb{R}^{d-1}$ $=$ $SO^{+}(1, d)$ $=$ isometry group of $\mathbb{H}^{d}$

The Dictionary

Notation: $G = \mathrm{SO}^+(1, d)$

a conformal field theory	$\longleftrightarrow$	a hyperbolic manifold $\Gamma \backslash \mathbb{H}^d$
Hilbert space	$\longleftrightarrow$	function space $L^2(\Gamma \backslash G)$
local operators	$\longleftrightarrow$	automorphic functions $F_i \in L^2(\Gamma \backslash G)$
correlation functions	$\longleftrightarrow$	$\langle F_1 \ldots F_n \rangle = \int_{\Gamma \backslash G} dg F_1(g) \ldots F_n(g)$
conformal Casimir eigenvalue	$\longleftrightarrow$	Laplacian eigenvalue $\lambda_i = \Delta_i(d - 1 - \Delta_i)$
operator product expansion	$\longleftrightarrow$	point-wise product $F_i(g) F_j(g) = \sum_k c_{ijk} F_k(g)$
structure constants	$\longleftrightarrow$	triple product integrals $c_{ijk} = \langle F_i F_j F_k \rangle$

Main Goals

Eventually: Extract lessons about interacting conformal field theories.

Today: Use techniques familiar in the conformal bootstrap to prove new results about the spectra of hyperbolic manifolds. We will focus on 2-manifolds.

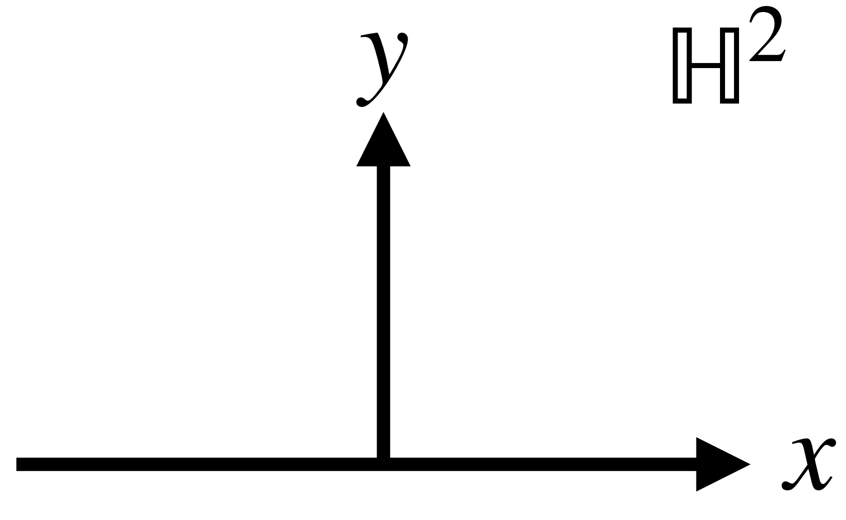
Previous Work

Bonifacio+Hinterbichler (2020): Einstein manifolds $R_{ab} = \frac{R}{d} g_{ab}$

Bonifacio (2021): Hyperbolic manifolds $R_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}$

Kravchuk, DM, Pal (2021): Pointed out the role played by $\mathrm{SO}(1,d)$ in the case of hyperbolic manifolds, and systematized the ideas using its representation theory.

2D Hyperbolic Orbifolds

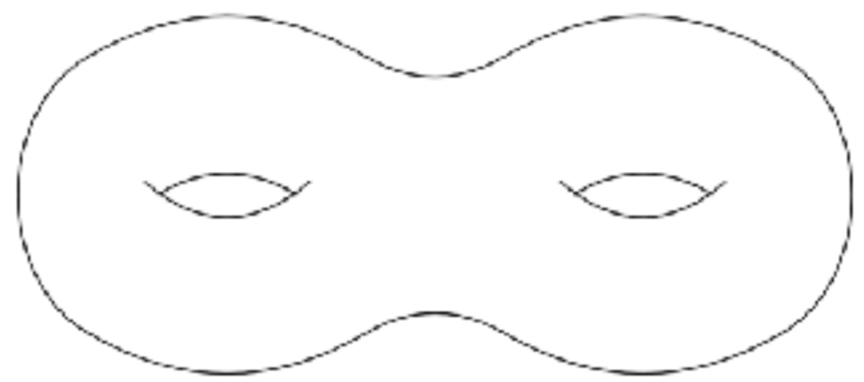


1. Upper half-plane with the hyperbolic metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$

   - $G = \mathrm{PSL}_2(\mathbb{R})$ acts on $z = x + iy \in \mathbb{H}^2$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : \; z \mapsto \dfrac{az+b}{cz+d}$

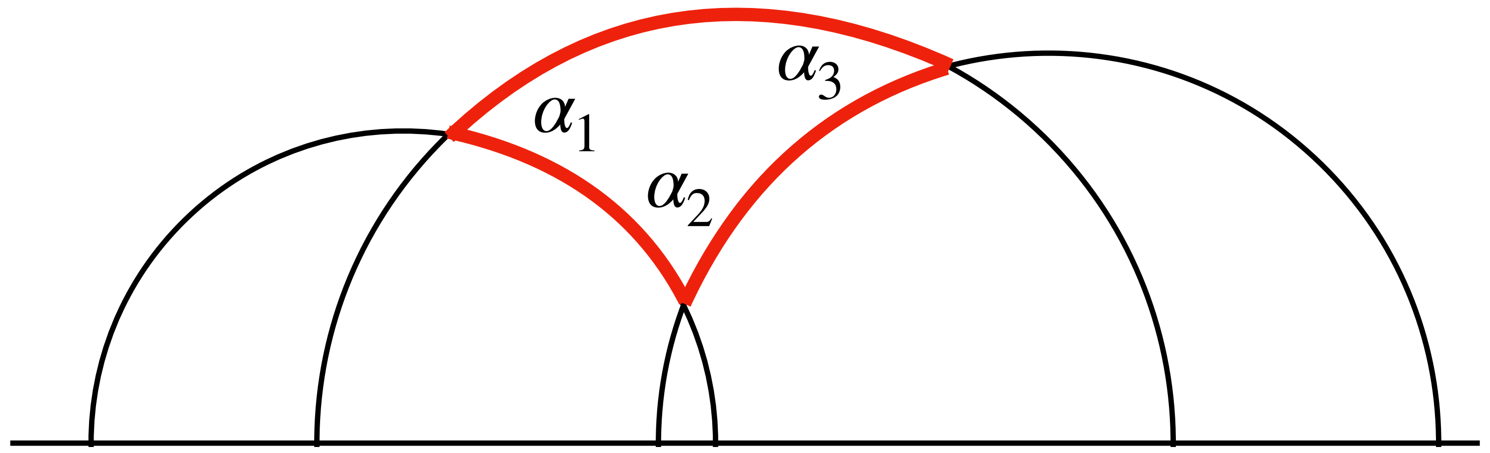
2. Γ = discrete subgroup of $\mathrm{PSL}_2(\mathbb{R}) \;\Leftrightarrow\; \Gamma \backslash \mathbb{H}^2$ = a hyperbolic orbifold.

$\Gamma \backslash \mathbb{H}^2$



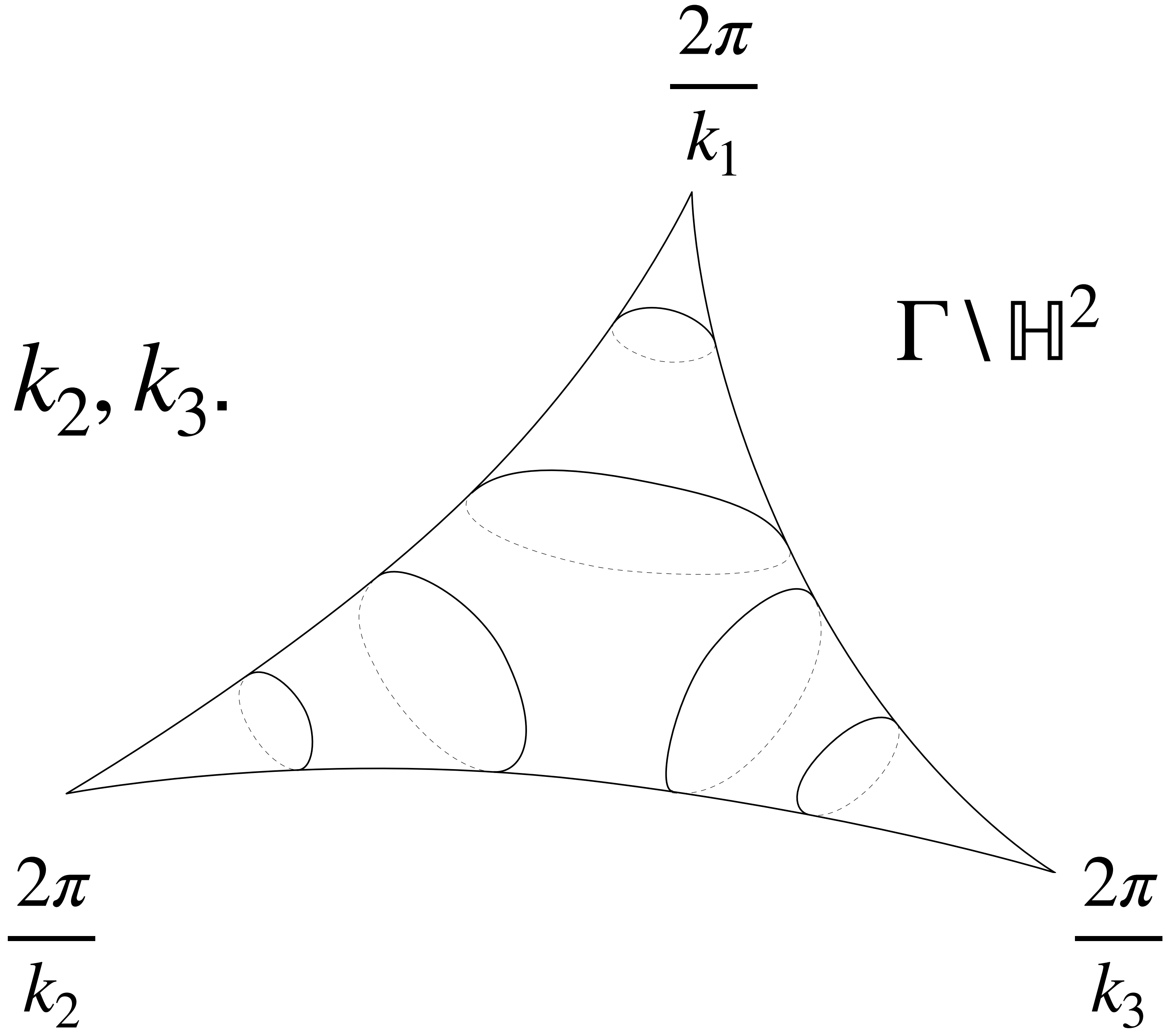
   - Will assume $\Gamma \backslash \mathbb{H}^2$ has finite volume.
   - Γ only has hyperbolic elements $\Leftrightarrow \Gamma \backslash \mathbb{H}^2$ is a compact surface.
   - Γ only has hyperbolic and elliptic elements $\Leftrightarrow \Gamma \backslash \mathbb{H}^2$ is a compact orbifold.

Example 1: Hyperbolic Triangle Groups



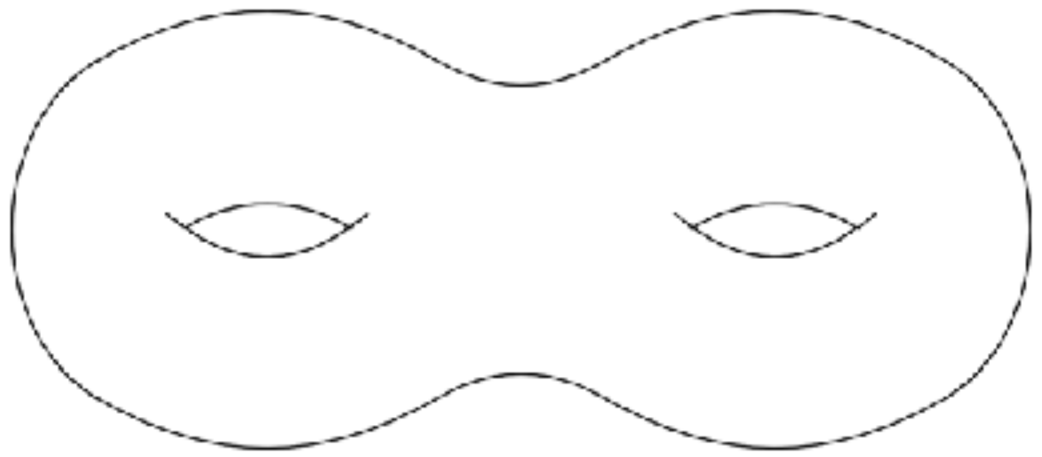
$$\alpha_i = \frac{\pi}{k_i} \qquad k_i \in \mathbb{N}_{\geq 2} \qquad \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} < 1 \;\Updownarrow\; \text{area} > 0$$

- Γ generated by rotations around vertices by angles $\dfrac{2\pi}{k_i}$.
- A fundamental domain of Γ consists of two adjacent triangles.
- $\Gamma \backslash \mathbb{H}^2$ is an orbifold of genus 0 with 3 orbifold points of orders k_1, k_2, k_3.
- Orbifold of minimal area: $[k_1, k_2, k_3] = [2,3,7]$.



Example 2: The Bolza Surface

A hyperbolic surface without orbifold points must have genus ≥ 2.

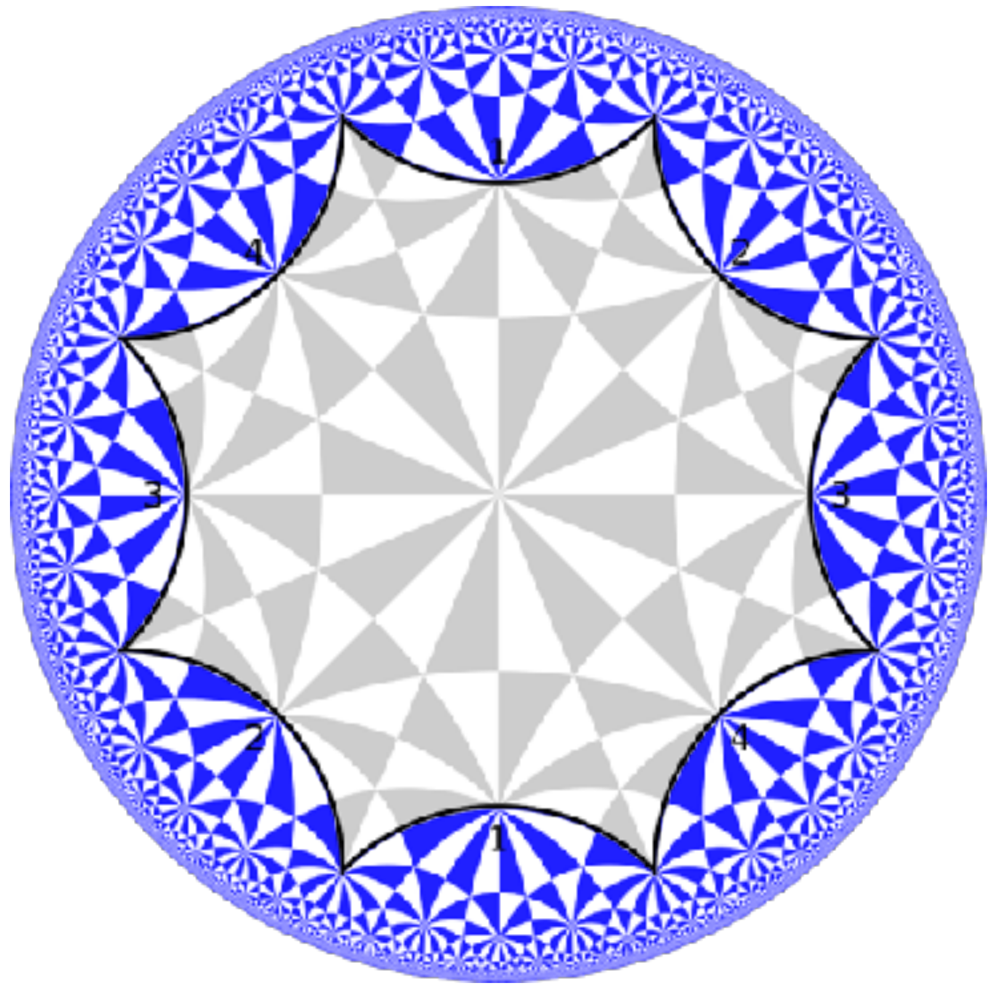


- Genus $= 2$: six-dimensional moduli space.

Bolza surface: the genus-two surface with the largest group of isometries.

- Iso(Bolza) = $\mathrm{GL}_2(\mathbb{F}_3)$, a group of order 48.

- Bolza = $\Gamma \backslash \mathbb{H}^2$, where Γ is a normal subgroup of index 48 of the [2,3,8] triangle group.



General Orbifolds

Topological type of $\Gamma \backslash \mathbb{H}^2$: $[g; k_1, \ldots, k_r]$ $\Leftrightarrow$ isomorphism type of Γ

genus orders of orbifold points

Laplacian Spectrum of $\Gamma \backslash \mathbb{H}^2$

The Laplacian on $\mathbb{H}^2$: $\nabla^2 = y^2(\partial_x^2 + \partial_y^2)$

$$-\nabla^2 h(x, y) = \lambda\, h(x, y)$$

$h(x, y)$: a smooth real function on $\mathbb{H}^2$ satisfying $h(\gamma \cdot (x, y)) = h(x, y)$ for all $\gamma \in \Gamma$.

Spectrum: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$

- no closed expression for λ_i in general
- a useful model for studying classical and quantum chaos

Today: New upper bounds on λ_1.

Main results

[Kravchuk, DM, Pal '21] [Bonifacio '21]

Theorem:

1. Every hyperbolic orbifold satisfies: $\lambda_1 \leq 44.8883537$.

   [2,3,7] triangle orbifold: $\lambda_1 \approx 44.88835$

2. Every hyperbolic orbifold of genus two satisfies: $\lambda_1 \leq 3.8388977$.

   Bolza surface: $\lambda_1 \approx 3.838887258$

   previous bound: $\lambda_1 \leq 4$ [Yang, Yau '80] [Soufi, Ilias '83]

3. Every hyperbolic orbifold of genus three satisfies: $\lambda_1 \leq 2.6784824$.

   Klein quartic: $\lambda_1 \approx 2.6779$

   previous bound: $\lambda_1 \leq 2(4 - \sqrt{7}) \approx 2.7085$ [Ros '20]

Spectrum of the Spectrum

Conjecture (Selberg 1965): If Γ is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then $\lambda_1 = 1/4$.

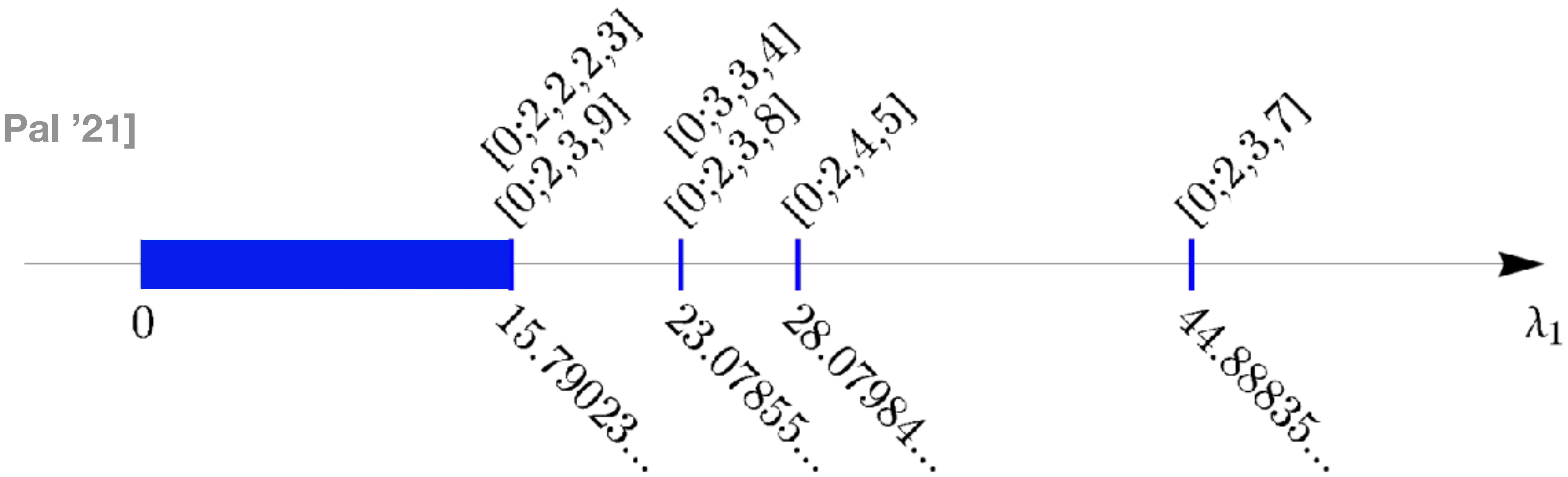
$$\Leftrightarrow$$

If X ranges over **congruence** orbifolds, the image of the map $X \mapsto \lambda_1(X)$ is the set $\{1/4\}$.

Question: What is the image of the map $X \mapsto \lambda_1(X)$ when X ranges over **all** orbifolds?

Answer:

[Kravchuk, DM, Pal '21]



The Method

1. The Hilbert space and local operators

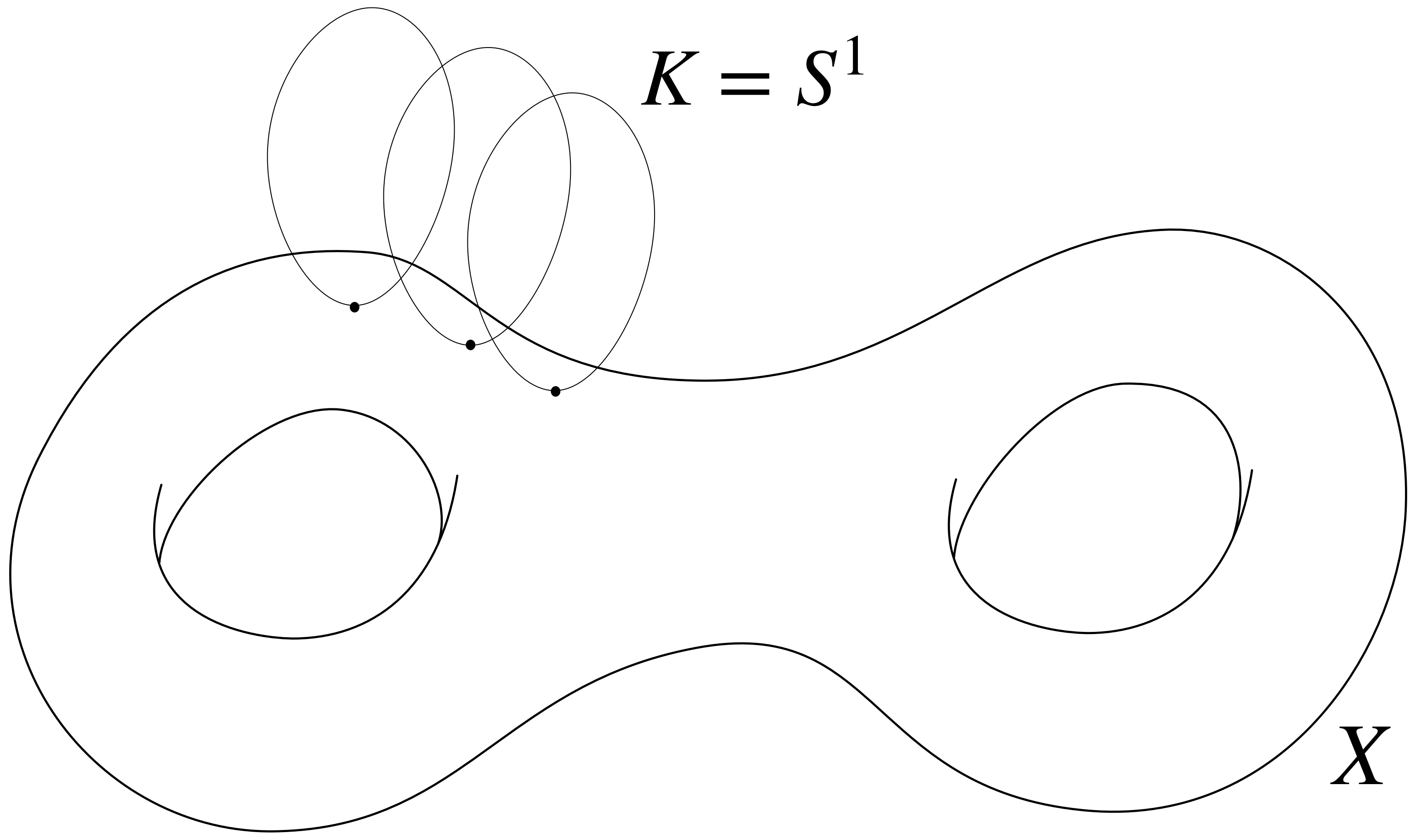
2. Operator product expansion

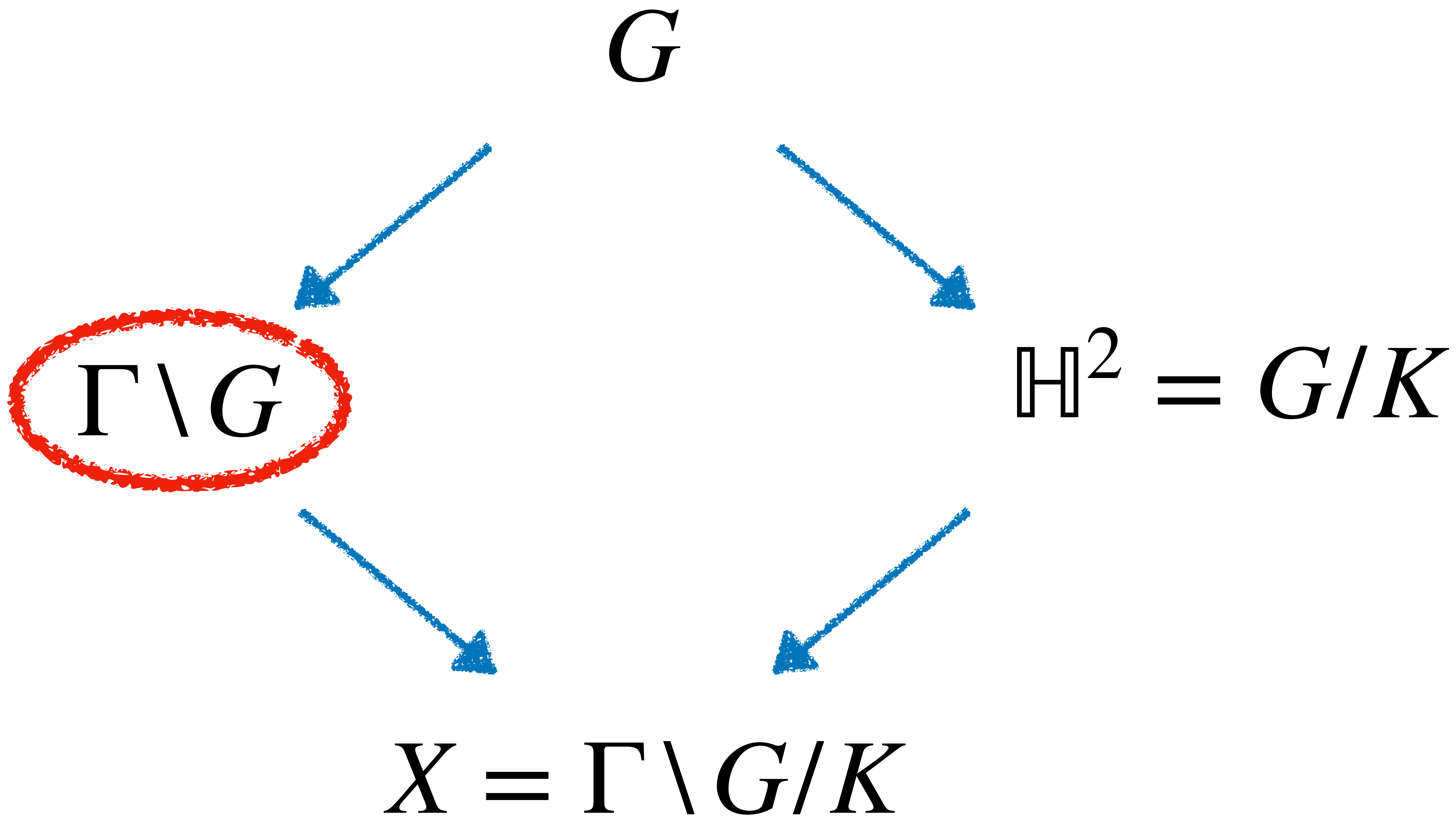
3. Associativity

4. Bounds from linear programming

The Coset Space

- $G = \mathrm{PSL}_2(\mathbb{R})$
- $K = \mathrm{PSO}_2(\mathbb{R})$, maximal compact subgroup of G
- Γ = discrete co-compact subgroup of G

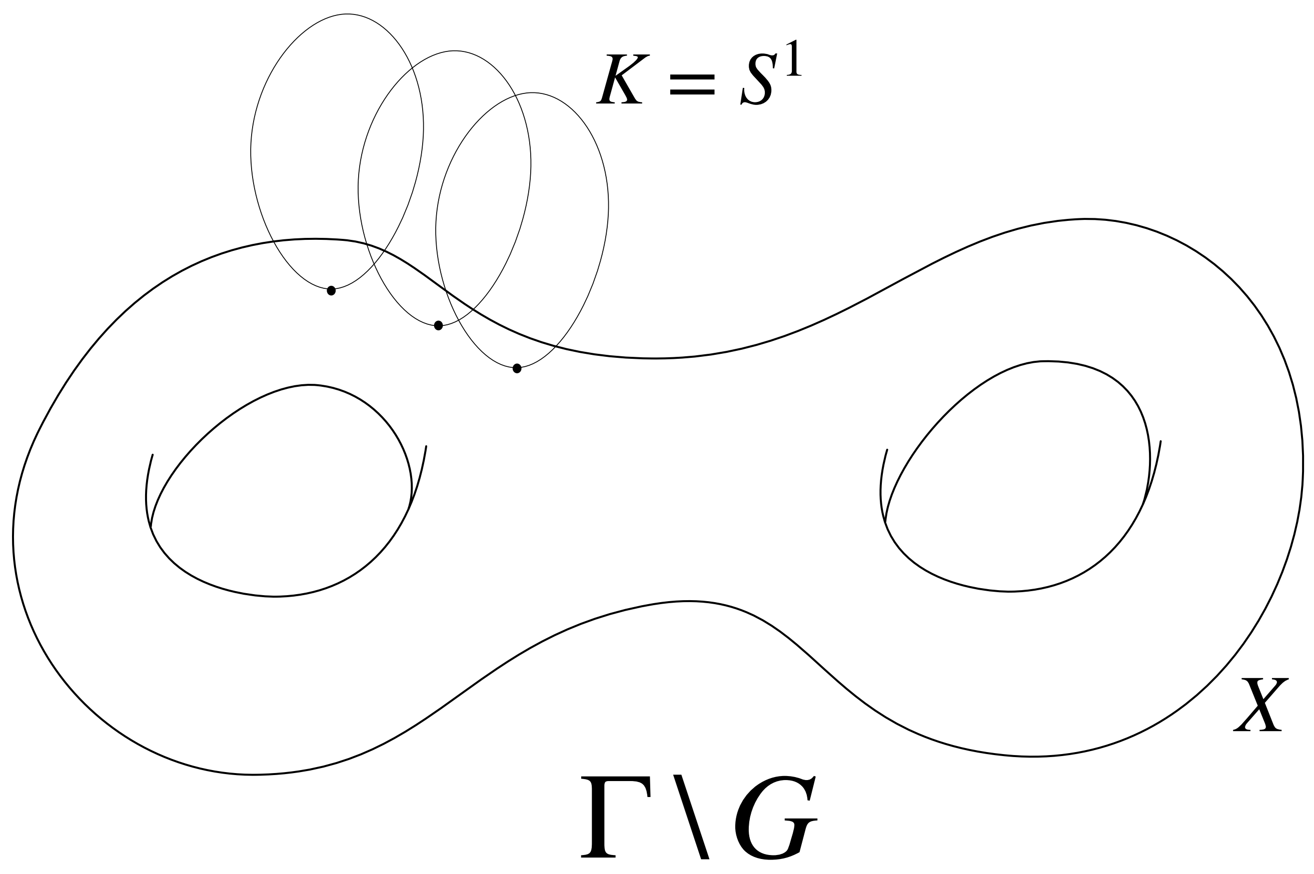




The Hilbert Space: $L^2(\Gamma \backslash G)$

Consider the space $L^2(\Gamma \backslash G)$

- a representation of G: $F(g) \overset{\tilde{g}}{\mapsto} F(g\tilde{g})$
- unitary, with inner product: $\|F(g)\|^2 = \int_{\Gamma \backslash G} dg \,|F(g)|^2$



Decomposition under K: $L^2(\Gamma \backslash G) = \bigoplus_{n \in \mathbb{Z}} V_n$

- $V_0 = L^2(X)$
- $V_n = L^2(n\text{-forms})$: $f(x, y)\, dz^n$ such that $\forall \gamma \in \Gamma$: $f(z) = (cz + d)^{-2n} f\left(\frac{az + b}{cz + d}\right)$
- Generators of G act as follows: $L_0|_{V_n} = n\,\mathrm{id}$, $L_{\pm 1} : V_n \to V_{n \mp 1}$

The Spectral Decomposition

Decompose $L^2(\Gamma \backslash G)$ into irreducible representations of $G = \mathrm{PSL}_2(\mathbb{R})$:

$$L^2(\Gamma \backslash G) = \mathbb{C} \oplus \bigoplus_{i=1}^{\infty} P_{\lambda_i} \oplus \bigoplus_{j=1}^{\infty} (D_{n_j} \oplus \overline{D}_{n_j})$$

1. Trivial representation $\mathbb{C}$: constant functions.

2. Principal and complementary series P_λ: Laplace eigenfunction with eigenvalue λ.
   - principal series: $\lambda \in [1/4, \infty)$, complementary series: $\lambda \in (0, 1/4)$.
   - Casimir$|_{V_0}$ = Laplacian $\Rightarrow v \in P_\lambda \cap V_0$ is a Laplace eigenfunction of eigenvalue λ.

3. Holomorphic discrete series D_n: holomorphic modular forms of weight $n \in \mathbb{N}_{>0}$.
   - $L_1 = \bar{\partial}$, $L_1|_{D_n \cap V_n} = 0 \Rightarrow v \in D_n \cap V_n$ is a holomorphic modular form of weight n.
   - Antiholomorphic discrete series $\overline{D}_n$: complex conjugates of modular forms.

Terminology: The Laplace eigenfunctions and holomorphic modular forms are examples of *automorphic forms.*

$$L^2(\Gamma\backslash G) = \mathbb{C} \oplus \bigoplus_{i=1}^{\infty} P_{\lambda_i} \oplus \bigoplus_{j=1}^{\infty} (D_{n_j} \oplus \overline{D}_{n_j})$$

Question: What are the constraints on the set of representations on the RHS?

Ingredients:

1. Riemann-Roch theorem: The topology of Γ determines the spectrum of holomorphic forms = discrete series. Namely, for $[g; k_1, \ldots, k_r]$, we have

$$\text{multiplicity}(D_n) = (2n-1)(g-1) + \sum_{i=1}^{r} \left\lfloor n\frac{k_i - 1}{k_i} \right\rfloor + \delta_{n,1}$$

$\Rightarrow$ Can focus on specific topology by making simple assumptions about the spectrum of D_n.

2. Consider the pointwise product $C^\infty(\Gamma\backslash G) \times C^\infty(\Gamma\backslash G) \to C^\infty(\Gamma\backslash G)$

$$(F_1(g), F_2(g)) \mapsto F_1(g)F_2(g)$$

Associativity and G-invariance $\Rightarrow$ bounds on the Laplacian spectrum.

Local Operators

Definition (local operator):

Let $F(g) \in L^2(\Gamma \backslash G)$ be a holomorphic modular form of weight n. Define

$$\mathcal{O}(w) = e^{wL_{-1}} \cdot F(g) = F(g) + wL_{-1} \cdot F(g) + \frac{w^2}{2} L_{-1}^2 \cdot F(g) + \ldots$$

Properties:

- $\mathcal{O}(w) \in L^2(\Gamma \backslash G) \cap D_n$ for $|w| < 1$.
- As w ranges over the unit disk, $\mathcal{O}(w)$ generates $L^2(\Gamma \backslash G) \cap D_n$.
- $\mathcal{O}(w)$ transforms like a conformal primary operator of scaling dimension n.

$$L_m \cdot \mathcal{O}(w) = [w^{m+1}\partial_z + (m+1)nw^m]\mathcal{O}(w)$$

Similarly, define the conjugate operator $\overline{\mathcal{O}}(w) = w^{-2n}e^{-L_1/w} \cdot \overline{F(g)}$.

- $\overline{\mathcal{O}}(w) \in L^2(\Gamma \backslash G) \cap \overline{D}_n$ for $|w| > 1$.

Correlation Functions

Definition (correlation function):

Given $F_1, \ldots, F_N \in C^\infty(\Gamma \backslash G)$, their correlation function is given by

$$\langle F_1 \ldots F_N \rangle = \frac{1}{\mathrm{vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} d\mu \, F_1(g) \ldots F_2(g)$$

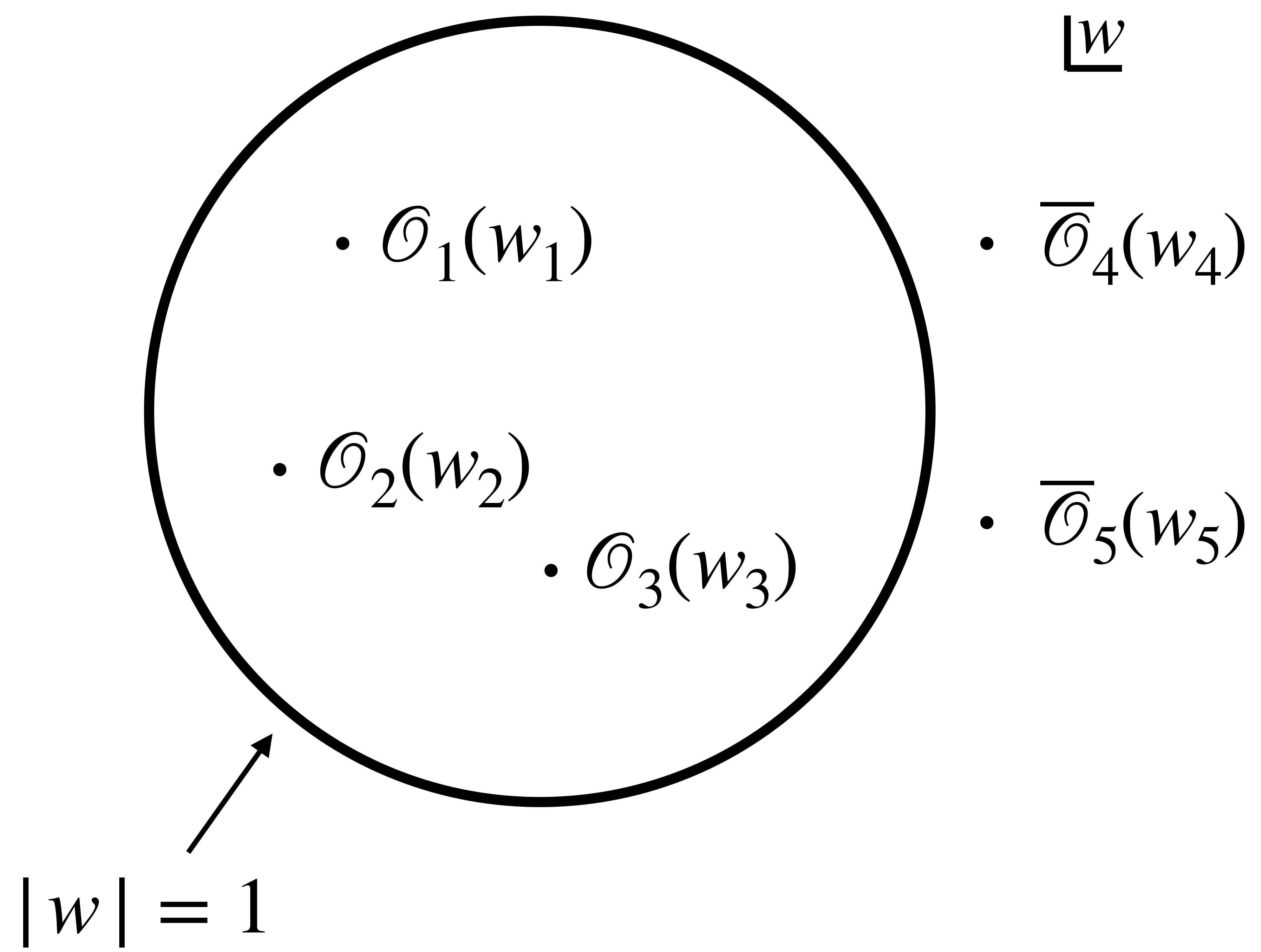
Since μ is G-invariant, so are the correlation functions.

Properties:

- one-point functions: $\langle 1 \rangle = 1, \langle \mathcal{O}_i(w) \rangle = \langle \overline{\mathcal{O}}_i(w) \rangle = 0$
- two-point functions: $\langle \mathcal{O}_i(w_1) \overline{\mathcal{O}}_j(w_2) \rangle = \dfrac{\delta_{ij}}{(w_1 - w_2)^{2n}}$

Each hyperbolic orbifold defines a large class of observables:

$$\langle \mathcal{O}_1(w_1)\ldots\mathcal{O}_N(w_N)\overline{\mathcal{O}}_{N+1}(w_{N+1})\ldots\overline{\mathcal{O}}_{N+M}(w_{N+M})\rangle$$



The Operator Product Expansion

Express products $\mathcal{O}(w_1)\overline{\mathcal{O}}(w_2)$, $\mathcal{O}(w_1)\mathcal{O}(w_2)$ using the spectral decomposition of $L^2(\Gamma \backslash G)$.

- $\mathcal{O}(w_1)\overline{\mathcal{O}}(w_2) = \dfrac{1}{(w_1 - w_2)^{2n}} + \sum_i c_i K_i(w_1, w_2)$, where $K_i(w_1, w_2) \in P_{\lambda_i}$.

- $\mathcal{O}(w_1)\mathcal{O}(w_2) = \sum_j \widetilde{c}_j \widetilde{K}_j(w_1, w_2)$, where $\widetilde{K}_j(w_1, w_2) \in D_{n_j}$.

Crucial fact: $K_i(w_1, w_2)$ and $\widetilde{K}_j(w_1, w_2)$ are universal = fixed by G-invariance.

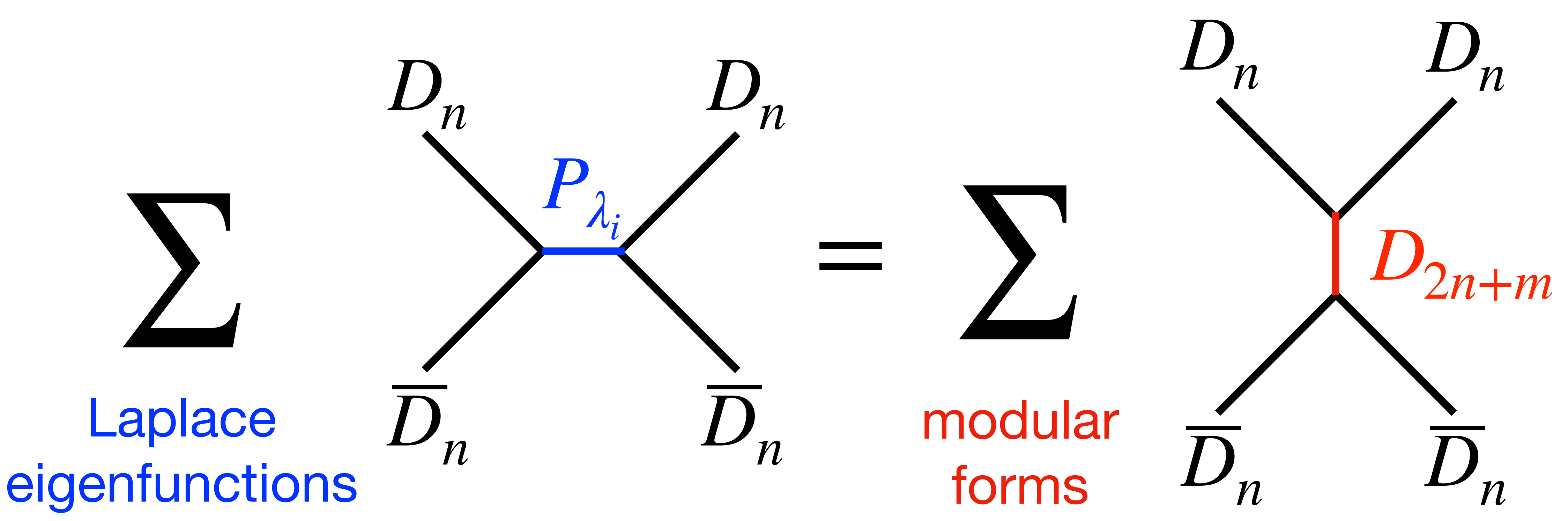
- The space of G-invariant maps $D_n \times \overline{D}_n \to P_\lambda$ and $D_n \times D_n \to D_m$ is one-dimensional.

- $c_i \sim \langle f \bar{f} h_i \rangle$, $\widetilde{c}_j \sim \langle f f \bar{f}_j \rangle$, integrals of triple products of automorphic forms.

Imposing Associativity

Suppose $L^2(\Gamma\backslash G)$ contains D_n and let $\mathcal{O}_n(w)$ be the corresponding local operator.

$$\langle \mathcal{O}_n(w_1)\mathcal{O}_n(w_2)\overline{\mathcal{O}}_n(w_3)\overline{\mathcal{O}}_n(w_4)\rangle$$



$$(1-\chi)^{-2n}\sum_i |c_i|^2 k_{\lambda_i}(\chi) = \chi^{-2n}\sum_{\substack{m\geq 0\\ m \text{ even}}} |\tilde{c}_m|^2 \tilde{k}_{2n+m}(\chi)$$

$$k_{s(1-s)}(\chi) = {}_2F_1(s,1-s;1;\frac{\chi}{\chi-1}) \qquad \tilde{k}_m(\chi) = \chi^m {}_2F_1(m,m;2m;\chi)$$

$\Rightarrow$ Get an infinite number of spectral identities by expanding around $\chi = 0$.

Spectral Bounds from Linear Programming

Spectral identities: $\sum_i |c_i|^2 P_{n,m}(\lambda_i) = |\widetilde{c}_m|^2$ for all even $m \geq 0$, $\sum_i |c_i|^2 P_{n,m}(\lambda_i) = 0$ for all odd $m > 0$

Proposition: Fix $M \in \mathbb{N}$ and suppose $Q(\lambda) = \sum_{m=0}^{M} x_m P_{n,m}(\lambda)$ with $x_m \in \mathbb{R}$, such that

1. $x_m \leq 0$ for all even m
2. $Q(0) = 1$
3. $Q(\lambda) \geq 0$ for all $\lambda \geq \lambda_*$.

Then there is an upper bound on the Laplace spectral gap $\lambda_1 < \lambda_*$ for every hyperbolic orbifold with a holomorphic form of weight n.

Proof: Consider $\sum_i |c_i|^2 Q(\lambda_i)$ and use the spectral identities. □

Strategy: Minimize λ_* by optimizing over x_m satisfying 1.-3. Increase M to improve the bound. We used the semidefinite programming solver SDPB. [Simmons-Duffin '15] [Simmons-Duffin, Landry '19]

Results

Let $n_1(\Gamma)$ be the minimal weight of a modular form for Γ.

Fact: We have $n_1(\Gamma) \in \{1, 2, 3, 4, 6\}$ for every hyperbolic orbifold.

n_1	our bound on λ_1	largest known λ_1	orbifold
1	8.47032	8.46776	$[1; 2]$ at the $\mathbb{Z}_6$-symmetric point
2	15.79144	15.79023	$[0; 2,2,2,3]$ at the $\mathbb{Z}_3$-symmetric point
3	23.07917	23.07855	$[0; 3,3,4]$
4	30.35432	28.07984	$[0; 2,4,5]$
6	44.8883537	44.88835	$[0; 2,3,7]$

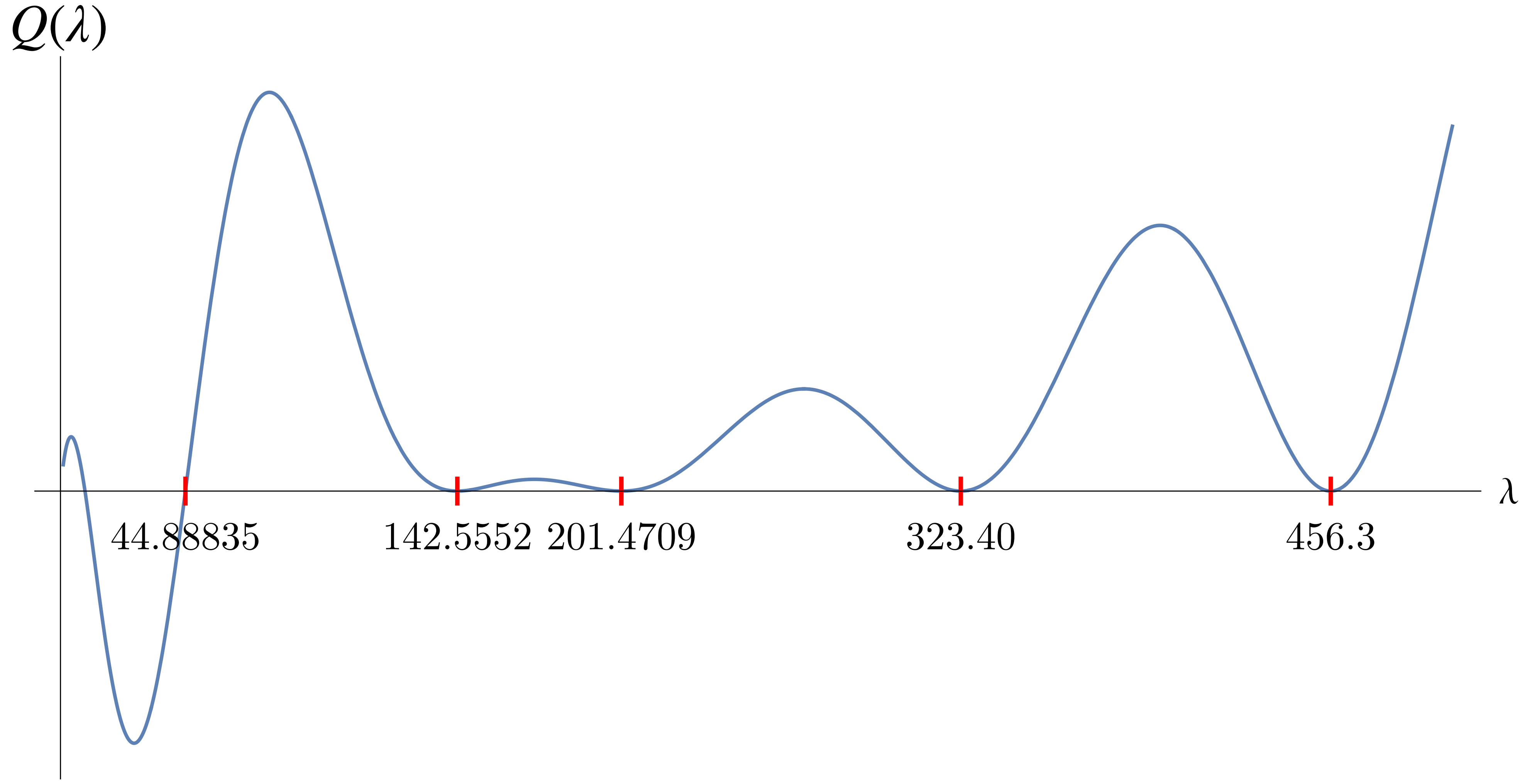
Corrolary: Every hyperbolic orbifold satisfies: $\lambda_1 \leq 44.8883537$.

Sharp Bounds

Question: Is the linear-programming upper bound on λ_1 sharp for $M \to \infty$?

If yes, the linear program must reconstruct the full Laplace spectrum of the $[0; 2,3,7]$ orbifold!

- $Q(\lambda_i) = 0$ for all $\lambda_i \in$ spectrum.
- Output of the linear program for $M = 41$ →
- Zeros agree with the $[0; 2,3,7]$ spectrum!
- Proof would amount to a construction of $Q(\lambda)$ for $M = \infty$.



This is precisely what happens for the Cohn-Elkies bound on sphere packing in $d = 8, 24$.

- Viazovska (2016): Construction of optimal $Q(\lambda)$ for sphere packing.
- DM (2016), DM+Paulos (2018): Construction of optimal $Q(\lambda)$ for the gap problem in 1D CFTs.
- Hartman+DM+Rastelli (2019): Precise mapping between Viazovska (2016) and DM (2016).

Challenge: Construct the optimal $Q(\lambda)$ for the Laplacian spectral gap problem.

Bounds at Fixed Genus

Bounds on λ_1 of genus-g orbifolds: Use g linearly independent holomorphic 1-forms.

Associativity implemented by the system of coupled equations:

$$\langle \mathcal{O}_i(w_1)\mathcal{O}_j(w_2)\overline{\mathcal{O}}_k(w_3)\overline{\mathcal{O}}_l(w_4)\rangle \qquad n_i = n_j = n_k = n_l = 1 \qquad i, j, k, l = 1, \ldots, g$$

This is a matrix generalization of the original linear program ⇒ need semidefinite programming.

genus	our bound on λ_1	largest known λ_1	orbifold
1	8.47032	8.46776	$[1;2]$ at the $\mathbb{Z}_6$-symmetric point
2	3.83890	3.83889	Bolza surface
3	2.67849	2.67793	Klein quartic

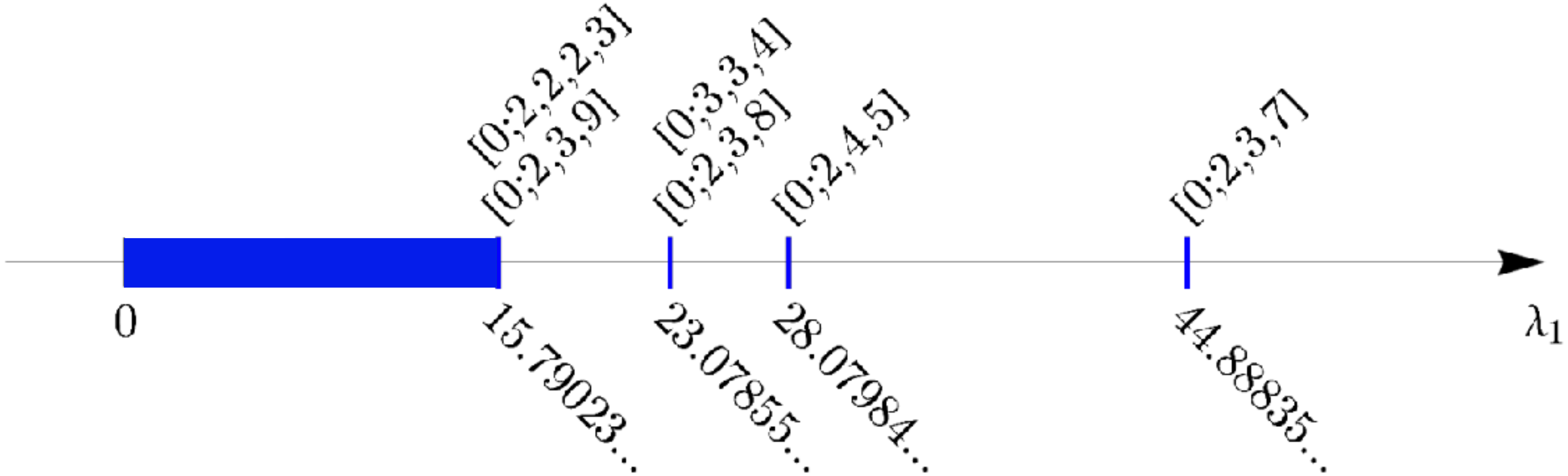
Values of λ_1 Attained by All Orbifolds

Idea: Topological type is uniquely identified by the spectrum of weights of modular forms.
Only finitely many weigths are needed to identify each topological type.

Study associativity for **two** holomorphic forms of minimal weight $1 \leq n_1 < n_2$

$$\langle \mathcal{O}_{n_1}(w_1)\mathcal{O}_{n_2}(w_2)\overline{\mathcal{O}}_{n_1}(w_3)\overline{\mathcal{O}}_{n_2}(w_4)\rangle$$

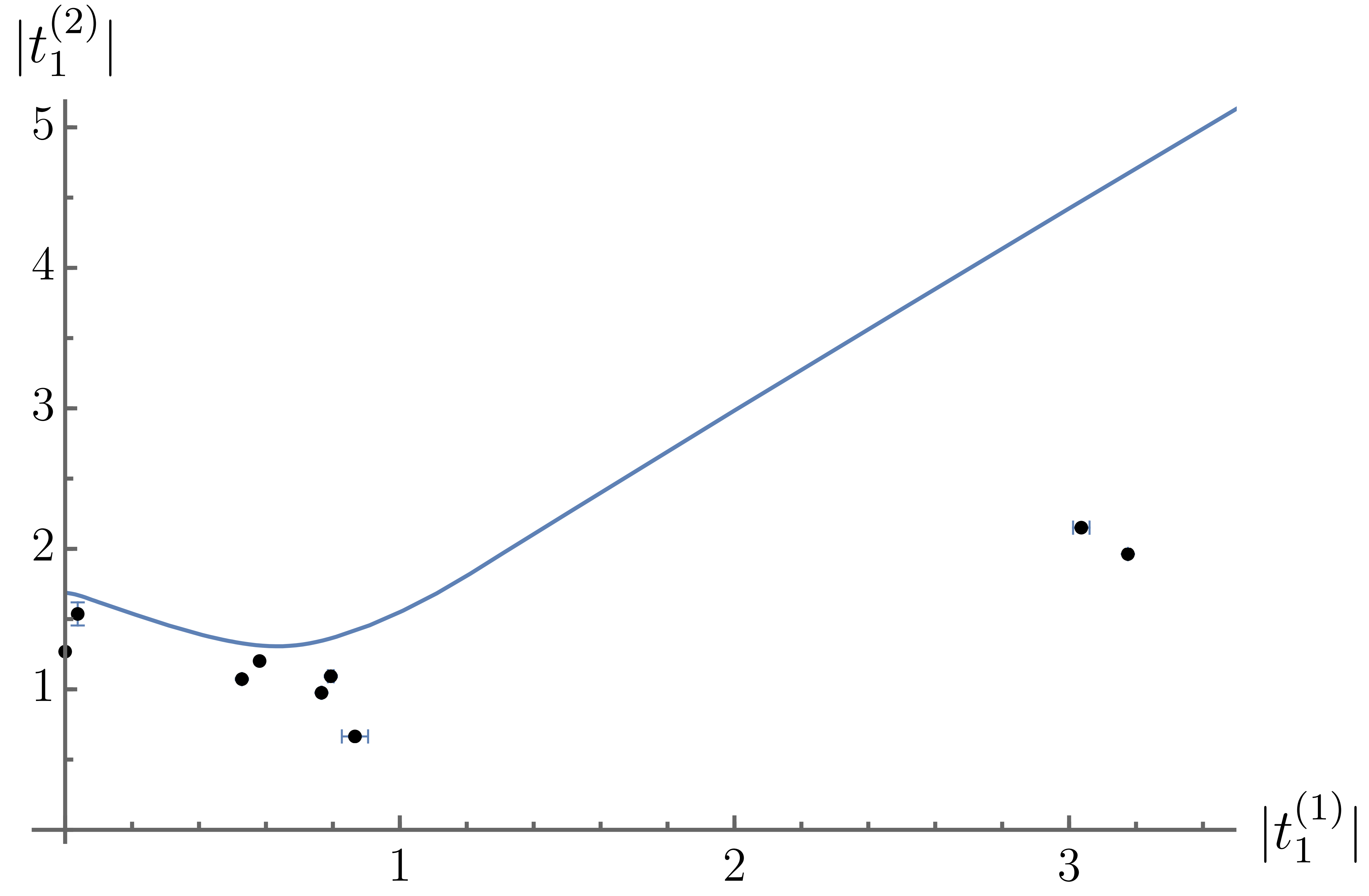
theorem: If X ranges over all orbifolds, $\lambda_1(X)$ takes the following values:



Example: $n_1 = 6, n_2 = 8 \Rightarrow \lambda_1 \leq 23.0997$ unless the orbifold is $[0; 2,3,7]$ or $n_1 \leq 4$.

Hyperbolic Three-Manifolds

work in progress with J. Bonifacio, P. Kravchuk and S. Pal



$|\, t_1^{(J)} \,|^2 + 1$ = the lowest Laplace eigenvalue on symmetric tensors of rank J.

Back to Conformal Field Theory

work in progress with J. Bonifacio, P. Kravchuk and S. Pal

Surprising finding: The same spectral identities apply to both hyperbolic manifolds and CFTs!

Justification:

- QFT path integral: measure space (Y, μ). $Y =$ space of field configurations in Eucl. space.
- Define the space of observables as $L^2(Y, \mu)$.
- Correlation functions: $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \int_Y d\mu \mathcal{O}_1 \ldots \mathcal{O}_n$, where $\mathcal{O}_i \in L^2(Y, \mu)$.
- CFT in $\mathbb{R}^d \Rightarrow \mathrm{SO}(1, d+1)$ acts on Y by measure-preserving transformations.

$\Rightarrow$ All the ingredients we used for hyperbolic manifolds are also present for CFTs.

- Checked the spectral identites are valid for 2D Ising CFT and generalized free theory.

The 3D Ising CFT and hyperbolic four-manifolds satisfy an infinite number of the same spectral identities.

Summary

- There is a close analogy between conformal field theories and hyperbolic manifolds.
- This leads to an infinite set of identities satisfied by the Laplacian spectra of hyp. manifolds.
- Linear/semidefinite programming turns the identites into bounds on the spectral gap λ_1.
- The bounds on λ_1 for 2D hyperbolic orbifolds are often nearly sharp.
- They allow us to (more or less) identify the set of λ_1 realized by all 2D hyperbolic orbifolds.

Future Directions

- Bounds on spectra of $d > 2$ hyperbolic manifolds.

- Bounds on triple overlaps $c_{ijk} = \int h_i h_j h_k \Leftrightarrow$ bounds on L-functions.

  [Sarnak]
  [Bernstein, Reznikov]
  [Michel, Venkatesh]
  [Nelson]

- Non-compact orbifolds, and the role of arithmeticity (Hecke operators), $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

- Extract lessons about the conformal bootstrap of CFTs.

- Spectral identites for QFT in de Sitter (the same symmetry and unitarity constraints).

Thank you!

The Dictionary

Notation: $G = \mathrm{SO}^+(1,d)$

a conformal field theory	$\longleftrightarrow$	a hyperbolic manifold $\Gamma \backslash \mathbb{H}^d$
Hilbert space	$\longleftrightarrow$	function space $L^2(\Gamma \backslash G)$
local operators	$\longleftrightarrow$	automorphic functions $F_i \in L^2(\Gamma \backslash G)$
correlation functions	$\longleftrightarrow$	$\langle F_1 \ldots F_n \rangle = \int_{\Gamma \backslash G} dg F_1(g) \ldots F_n(g)$
conformal Casimir eigenvalue	$\longleftrightarrow$	Laplacian eigenvalue $\lambda_i = \Delta_i(d-1-\Delta_i)$
operator product expansion	$\longleftrightarrow$	point-wise product $F_i(g)F_j(g) = \sum_k c_{ijk} F_k(g)$
structure constants	$\longleftrightarrow$	triple product integrals $c_{ijk} = \langle F_i F_j F_k \rangle$