From BPS crystals to BPS algebras

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based on joint work with Masahito Yamazaki and Dmitrii Galakhov

BPS algebra

Question: What is the algebraic structure underlying the BPS sector of a 4D $\mathcal{N}=2$ theory?

Definition: BPS algebra (algebra of the BPS states) *Harvey-Moore '96*

$$\boxed{\text{multiplication:} \qquad \mathcal{H}_{\text{BPS}} \otimes \mathcal{H}_{\text{BPS}} \to \mathcal{H}_{\text{BPS}}}$$

- Analogue of chiral algebra of 2D $\mathcal{N}=2$ SCFT
- Robust and control many aspects of theory (BPS counting, wall-crossing ...)

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Today: Type IIA string on a generic toric CY_3

- $\frac{1}{2}$-BPS sector: D6/D4/D2/D4 wrapping holomorphic cycles

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BPS algebra for string on toric CY_3

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Kontsevich-Soibelman '10

cf. Yaping's talk on Monday

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  [$\mathbb{C}^3$]

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- (truncations of) $\mathcal{W}$ algebras [without compact 4-cycle] — AGT, "Corner VOA"

  Alday-Gaiotto-Tachikawa '09

  Gaiotto-Rapčák '17, Rapčák-Prochàzka '18

  Eberhardt-Prochàzka '19, Rapčák '19

Relations among different formulations (e.g. $\mathbb{C}^3$)

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  [without compact 4-cycle]

Today: will derive affine Yangian formulation of the BPS algebra

- based on how they act on BPS states
- applies to all toric Calabi-Yau threefolds
- easy to write down explicit relations

BPS quiver Yangians from colored crystals

L-Yamazaki '20

Type IIA string on a generic toric CY_3

1. $\frac{1}{2}$-BPS sector: $\mathcal{N}=4$ quiver QM (Q,W)

   $\Downarrow$ define

2. { BPS states } = { 3d colored crystals }

   act $\Uparrow \Downarrow$ bootstrap

3. BPS algebra = quiver Yangian $\mathsf{Y}(Q,W)$

Advantages

1. apply to any toric Calabi-Yau threefolds
2. explicit algebraic relations
3. easy to generalized to trigonometric and elliptic versions
4. easy to describe representations
   (corresponding to different chambers and can include open BPS)

Outline

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Worldvolume theory on D-brane bound state

IIA string on a toric CY_3 X

- $\frac{1}{2}$-BPS sector with D6/D4/D2/D0 brane on holomorphic 6/2/0 cycles of X

$$\text{First, consider} \qquad \#(\mathrm{D6},\ \mathrm{D4},\ \mathrm{D2},\ \mathrm{D0}) = (1, 0, m_i, n)$$

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toric data $X \longrightarrow$ quiver data (Q, W) (Brane tiling: *Hanany Kennaway Vegh ...*)

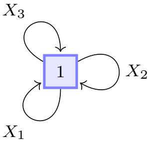
1. Quiver $Q = (Q_0, Q_1)$

$$Q_0 = \{\text{vertex } a\} \qquad a : \; U(N_a) \text{ gauge group}$$

$$Q_1 = \{\text{arrow } I : a \to b\} \qquad \Phi_I : \; \text{bi-fundamentals } (\overline{N_a}, N_b)$$

2. superpotential $W = \sum \pm \prod \Phi_I$ (with each Φ_I appearing twice with $\pm$)

$\mathbb{C}^3$:



$$W = \mathrm{Tr}[-X_1\, X_2\, X_3 + X_1\, X_3\, X_2]$$

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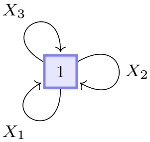
$$(Q, W) \Longleftrightarrow \text{periodic } \tilde{Q} = (Q_0, Q_1, Q_2) \ \text{ on } \ \mathbb{R}^2$$

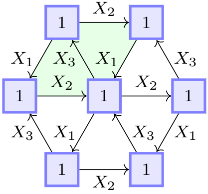
(each face in $Q_2 \leftrightarrow$ a term in W with $\pm$)

$\mathbb{C}^3$:



$$W = \mathrm{Tr}[-X_1 X_2 X_3 + X_1 X_3 X_2]$$

$\Longleftrightarrow$



Each D-brane bound state $\leftrightarrow$ a crystal configuration

Szendröi '07, Mozgovoy-Reineke '07, Ooguri-Yamazaki '08

a D-brane bound state with charge $(1, 0, m_j, n)$ in toric CY_3 X

$$\Updownarrow$$

a $U(1)^2$-inv. solution (of F/D-term) in quiver QM (Q, W) with rank $\{N_a\}$

$$\Updownarrow$$

a 3D crystal K (uplifted from periodic quiver $\tilde{Q}$) with $\{N_a$ number of $\boxed{a}\}$

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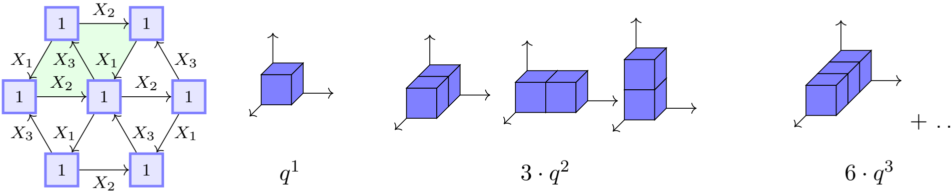

$$Z_{\text{crystal}}(q) = 1 + q + 3\,q^2 + 6\,q^3 + 13\,q^4 + \cdots = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}$$

$$= Z_{\text{BPS}}(q)$$

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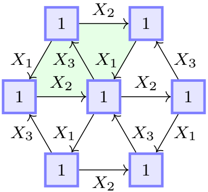

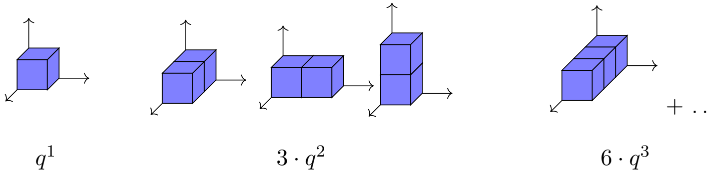

For $\mathbb{C}^3$, no D2 brane and number of D0 = number of boxes in plane partition

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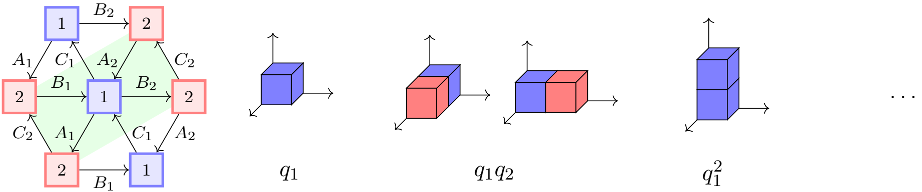
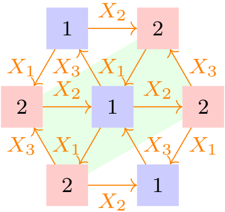


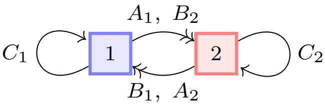
In general, number of D2 and D0 $\longrightarrow$ number of colored atoms in crystal

$$Z_{\text{crystal}} = Z_{\text{BPS}}$$

BPS crystal from uplifting periodic quiver: $(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$



$\Leftarrow$



$$W = \mathrm{Tr}[-C_m\, A_m\, B_m + C_m\, B_{m+1}\, A_{m+1}]$$

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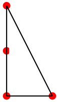
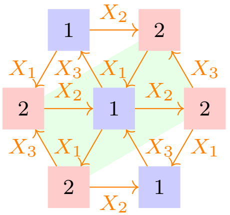


Framing (consider NCDT chamber first)

1. set framed vertex $a_{\mathfrak{f}} = 1$



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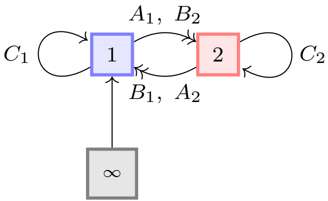
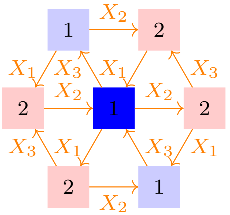


Origin of crystal

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in periodic quiver



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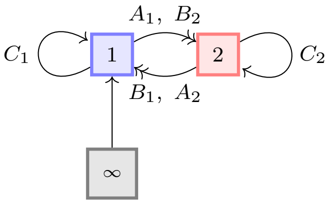
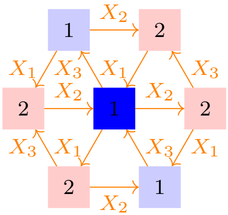


path $\Rightarrow$ atom

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in periodic quiver

2. path from $\mathfrak{o}$ $\Rightarrow$ atom $\boxed{a}$



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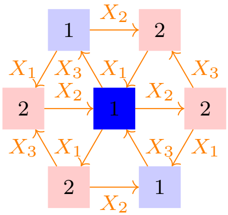

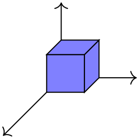

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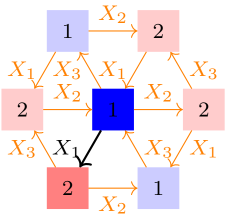

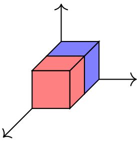

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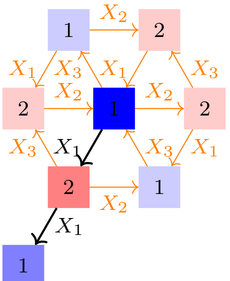
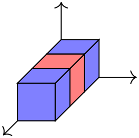
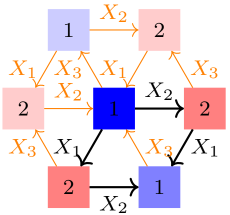
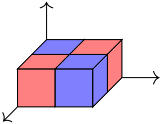




Path equivalence (from F-term constraint)

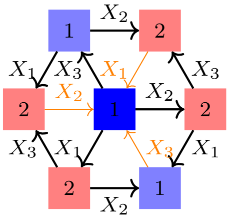
1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in periodic quiver
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. path equivalence

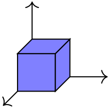




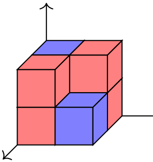
Depth of an atom

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in periodic quiver
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. path equivalence
4. depth $=$ number of closed loop in the path





depth $= 0$



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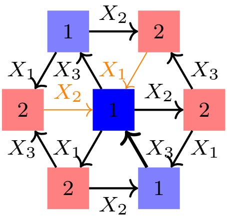

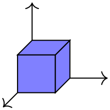

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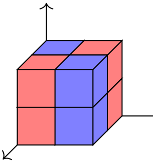
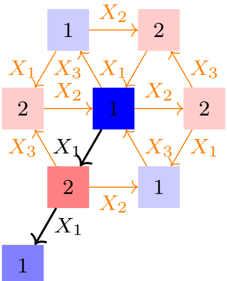
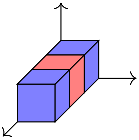


depth $= 1$

Adding rule (a.k.a. melting rule)

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in periodic quiver
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. path equivalence
4. depth $=$ number of closed loop in the path
5. Adding rule: to add an atom, all its precusors have to be already in the crystal





allowed

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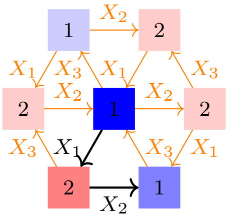

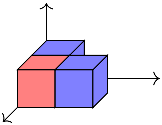

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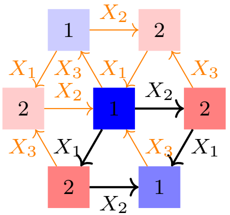

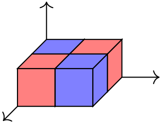

allowed

Equivariant weights of arrows and atoms

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in periodic quiver
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$ $\Longrightarrow$ $h(\boxed{a}) = \sum_{I \in \text{path}[\mathfrak{o} \to \boxed{a}]} h_I$
3. path equivalence $\Longrightarrow$ Loop constraint $\sum_{I \in L} h_I = 0$
4. depth $=$ number of closed loop in the path
   projection: same $h(\boxed{a})$ with different depth
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> To derive BPS algebra from crystal, turn on Ω background
> and assign equivariant weights to atoms

L-Yamazaki '20

1. h_I: equivariant weight of arrow I $\Longrightarrow$ $h(\boxed{a})$: equivariant weight of atom $\boxed{a}$

Equivariant weights of arrows and atoms

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in periodic quiver

2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$ $\Longrightarrow$ $h(\boxed{a}) = \sum_{I \in \text{path}[\mathfrak{o} \to \boxed{a}]} h_I$

3. path equivalence $\Longrightarrow$ Loop constraint $\sum_{I \in L} h_I = 0$

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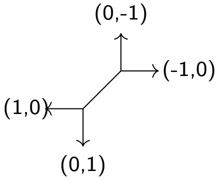
> To derive BPS algebra from crystal, turn on Ω background and assign equivariant weights to atoms

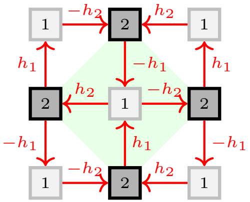
L-Yamazaki '20

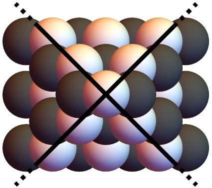
1. h_I: equivariant weight of arrow I $\Longrightarrow$ $h(\boxed{a})$: equivariant weight of atom $\boxed{a}$
2. Also impose vertex constraint $\sum_{I \in a} \text{sign}_a(I)\, h_I = 0$ (gauge symmetry)
3. After loop and vertex constraints, the number of parameters=2

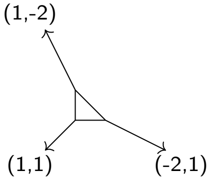
Toric CY_3 $\Longrightarrow$ periodic quiver $\Longrightarrow$ 3D crystal



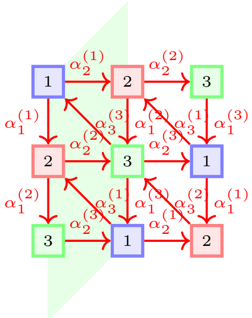
conifold

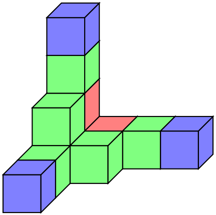






$\mathbb{C}^3/\mathbb{Z}_3$





Outline

Bootstrapping BPS algebra

BPS algebra acts on $\mathcal{H}_{\text{BPS}}$ and reproduces fusion process:

$$\begin{aligned} \text{BPS algebra} \cdot \mathcal{H}_{\text{BPS}} &\to \mathcal{H}_{\text{BPS}} \\ \alpha \;\; |\text{K}_1\rangle \;\; &= |\text{K}_2\rangle \end{aligned}$$

1. Define generators as changing BPS state (crystal) by smallest unit (atom)

$$\begin{cases} \text{raising} \quad e^{(a)}: & |\text{K}\rangle \to |\text{K} + \boxed{a}\rangle \\ \text{Cartan} \quad \psi^{(a)}: & |\text{K}\rangle \to |\text{K}\rangle \\ \text{lowering} \quad f^{(a)}: & |\text{K}\rangle \to |\text{K} - \boxed{a}\rangle \end{cases}$$

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2. Fix actions of (e, ψ, f) on any crystal $|\text{K}\rangle$
3. Find all relations of (e, ψ, f) on any crystal $|\text{K}\rangle$ $(F(e, \psi, f)|\text{K}\rangle = 0)$

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$$\text{BPS algebra} = \frac{< e, \psi, f >}{\{F(e, \psi, f) = 0\}}$$

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Expect $(e_i^{(a)}, \psi_i^{(a)}, f_i^{(a)})$ ∞−dim

$$e^{(a)}(z) \equiv \sum_j \frac{e_j^{(a)}}{z^{j+1}} \qquad \psi^{(a)}(z) \equiv \sum_j \frac{\psi_j^{(a)}}{z^{j+1}} \qquad f^{(a)}(z) \equiv \sum_j \frac{f_j^{(a)}}{z^{j+1}} \quad a \in Q_0$$

Ansatz for action of BPS algebra on BPS crystal *L-Yamazaki '20*

$$\begin{cases} \text{Cartan:} & \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \\ \text{raising:} & e^{(a)}(z)|\mathrm{K}\rangle = \sum \frac{E(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \\ \text{lowering:} & f^{(a)}(z)|\mathrm{K}\rangle = \sum \frac{F(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \end{cases}$$

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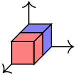
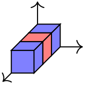
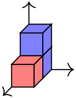
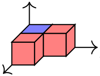
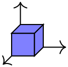
Need to define Ψ_{K}, E and F such that E and F are non-zero only when $|\mathrm{K}\pm\boxed{a}\rangle$ are valid crystal states.

- Example: $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$

$|\mathrm{K}\rangle =$  $\Longrightarrow$ $|\mathrm{K}+\boxed{a}\rangle = \{$ , ,  $\}$

$|\mathrm{K}-\boxed{a}\rangle =$ 

Ansatz for action of BPS algebra on BPS crystal *L-Yamazaki '20*

$$
\begin{cases}
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\text{lowering:} & f^{(a)}(z)|\mathrm{K}\rangle = \sum \dfrac{F(\mathrm{K} \to \mathrm{K} - \boxed{a})}{z - h(\boxed{a})}|\mathrm{K} - \boxed{a}\rangle
\end{cases}
$$

Need to define Ψ_{K}, E and F such that E and F are non-zero only when $|\mathrm{K} \pm \boxed{a}\rangle$ are valid crystal states.

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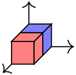
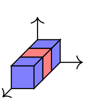
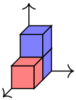
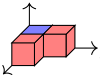
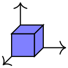
$|\mathrm{K}\rangle =$  $\Longrightarrow$ $|\mathrm{K} + \boxed{a}\rangle = \{$  ,  ,  $\}$

$|\mathrm{K} - \boxed{a}\rangle =$ 

How to realize this?

Ansatz for action of BPS algebra on BPS crystal *L-Yamazaki '20*

($\mathbb{C}^3$, affine Yangian of $\mathfrak{gl}_1$: *Tsymbaliuk '14, Prochazka '15*)

$$\begin{cases} \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi_{\mathrm{K}}^{(a)}(z)|\mathrm{K}\rangle\,, \qquad \Psi_{\mathrm{K}}^{(a)}(u) \equiv \psi_0^{(a)}(z) \prod_{\boxed{b}\in \mathrm{K}} \varphi^{a\Leftarrow b}(u - h(\boxed{b})) \\ e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Add}(\mathrm{K})} \frac{\pm\sqrt{\mathrm{Res}_{u=h(\boxed{a})}\Psi_{\mathrm{K}}^{(a)}(u)}}{z - h(\boxed{a})} |\mathrm{K}+\boxed{a}\rangle\,, \\ f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Rem}(\mathrm{K})} \frac{\pm\sqrt{(-1)^{|a|}\mathrm{Res}_{u=h(\boxed{a})}\Psi_{\mathrm{K}}^{(a)}(u)}}{z - h(\boxed{a})} |\mathrm{K}-\boxed{a}\rangle\,, \end{cases}$$

Ansatz for action of BPS algebra on BPS crystal *L-Yamazaki '20*

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$$\text{Demand:} \qquad \{\text{poles of } \Psi^{(a)}_{\mathrm{K}}(z)\} \quad \simeq \quad \mathrm{Add}^{(a)}(\mathrm{K}) \cup \mathrm{Rem}^{(a)}(\mathrm{K})$$

1. Provide all the necessary poles.
2. No redundant pole ("Adding Rule" is satisfied automatically)

Ansatz for action of BPS algebra on BPS crystal *L-Yamazaki '20*

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Demand: $\quad \{\text{poles of } \Psi^{(a)}_{\mathrm{K}}(z)\} \quad \simeq \quad \mathrm{Add}^{(a)}(\mathrm{K}) \cup \mathrm{Rem}^{(a)}(\mathrm{K})$

ground state: $\quad \psi^{(a)}_0(z) = \left(\frac{1}{z}\right)^{\delta_{a,1}} \quad$ (NCDT chamber)

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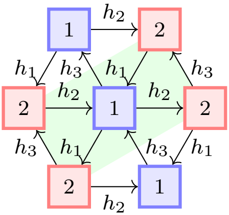
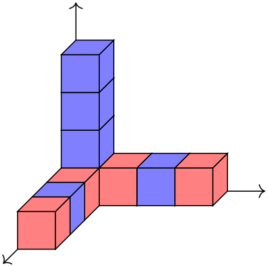
Demand: $\{\text{poles of } \Psi^{(a)}_{\mathrm{K}}(z)\} \simeq \mathrm{Add}^{(a)}(\mathrm{K}) \cup \mathrm{Rem}^{(a)}(\mathrm{K})$

Bonding factor: $\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}} \frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)}$

Example: $(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$

- Loop constraint: $h_1 + h_2 + h_3 = 0$



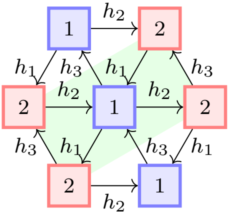


Example: $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$

- Loop constraint: $h_1+h_2+h_3=0$
- building blocks of $\Psi^{(a)}_{\mathrm{K}}(u)$

$$\psi_0^{(1)}(u)=\frac{1}{z}\qquad \varphi^{1\Leftarrow 1}(u)=\frac{u+h_3}{u-h_3}\qquad \varphi^{1\Leftarrow 2}(u)=\frac{(u+h_1)(u+h_2)}{(u-h_1)(u-h_2)}$$

$$\psi_0^{(2)}(u)=1\qquad \varphi^{2\Leftarrow 1}(u)=\frac{(u+h_1)(u+h_2)}{(u-h_1)(u-h_2)}\qquad \varphi^{2\Leftarrow 2}(u)=\frac{u+h_3}{u-h_3}$$



- Ground state contribution:

$$\psi_0^{(a)}(z)=\left(\frac{1}{z}\right)^{\delta_{a,1}}\qquad \text{(NCDT chamber)}$$

- Bonding factor:

$$\varphi^{a\Leftarrow b}(u)\equiv(-1)^{|b\to a|\chi_{ab}}\frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)}$$

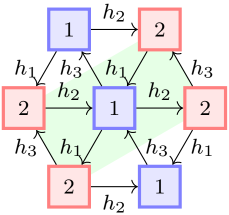
Example: $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$

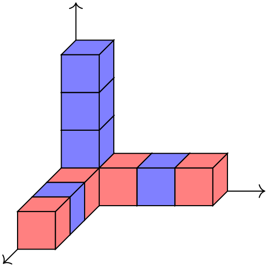
- Loop constraint: $h_1 + h_2 + h_3 = 0$
- building blocks of $\Psi_{\mathrm{K}}^{(a)}(u)$

$$\psi_0^{(1)}(u) = \frac{1}{z} \qquad \varphi^{1\Leftarrow 1}(u) = \frac{u+h_3}{u-h_3} \qquad \varphi^{1\Leftarrow 2}(u) = \frac{(u+h_1)(u+h_2)}{(u-h_1)(u-h_2)}$$

$$\psi_0^{(2)}(u) = 1 \qquad \varphi^{2\Leftarrow 1}(u) = \frac{(u+h_1)(u+h_2)}{(u-h_1)(u-h_2)} \qquad \varphi^{2\Leftarrow 2}(u) = \frac{u+h_3}{u-h_3}$$

- Check cyclicity (i.e. all necessary poles) and adding rule (i.e. no redundant pole)

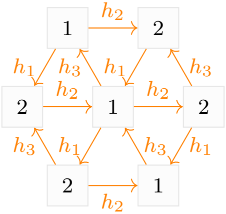


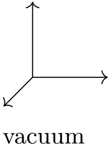


$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: vacuum

1. vacuum $|\emptyset\rangle$
2. Charge functions

$$\begin{cases} \Psi^{(1)}_{\mathrm{K}}(z) = \psi^{(1)}_0(z) = \dfrac{1}{z} \\ \end{cases}$$





$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: vacuum

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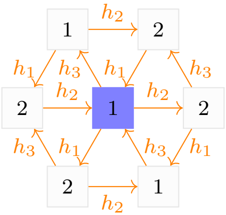
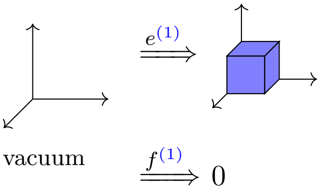
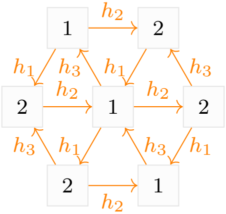


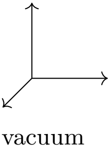


$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: vacuum

1. vacuum $|\emptyset\rangle$
2. Charge functions

$$\begin{cases} \Psi_{\mathrm{K}}^{(1)}(z) = \psi_0^{(1)}(z) = \dfrac{1}{z} \\ \Psi_{\mathrm{K}}^{(2)}(z) = \psi_0^{(2)}(z) = 1 \end{cases}$$





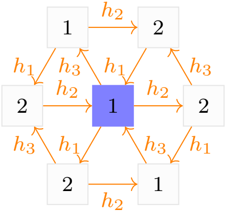
$\xRightarrow{e^{(2)}} 0$

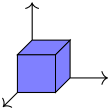
$\xRightarrow{f^{(2)}} 0$

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: level-1

1. 1-atom state $|\boxed{1}\rangle$, with $h(\boxed{1}) = 0$
2. Charge functions

$$\begin{cases} \Psi_{\mathrm{K}}^{(1)}(z) = \psi_0^{(1)}(z) \cdot \varphi^{1 \Leftarrow 1}(z - h(\boxed{1})) = \frac{1}{z} \cdot \frac{z + h_3}{z - h_3} \\ \end{cases}$$

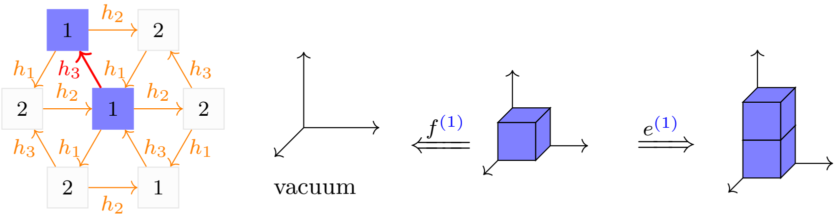




$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: level-1

1. 1-atom state $|\boxed{1}\rangle$, with $h(\boxed{1}) = 0$
2. Charge functions

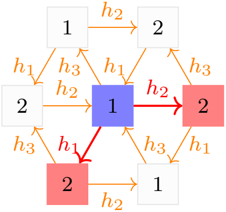
$$\begin{cases} \Psi_{\mathrm{K}}^{(1)}(z) = \psi_0^{(1)}(z) \cdot \varphi^{1\Leftarrow 1}(z - h(\boxed{1})) = \dfrac{1}{z} \cdot \dfrac{z + h_3}{z - h_3} \\ \end{cases}$$

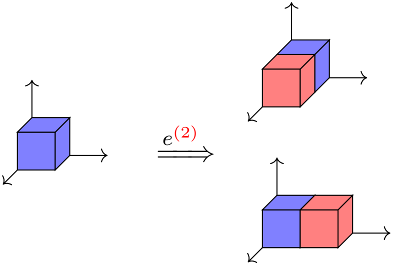


$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-1

1. 1-atom state $|\boxed{1}\rangle$, with $h(\boxed{1})=0$
2. Charge functions

$$\begin{cases} \Psi_{\mathrm{K}}^{(1)}(z)=\psi_0^{(1)}(z)\cdot\varphi^{1\Leftarrow 1}(z-h(\boxed{1}))=\dfrac{1}{z}\cdot\dfrac{z+h_3}{z-h_3} \\ \Psi_{\mathrm{K}}^{(2)}(z)=\psi_0^{(2)}(z)\cdot\varphi^{2\Leftarrow 1}(z-h(\boxed{1}))=\dfrac{(z+h_1)(z+h_2)}{(z-h_1)(z-h_2)} \end{cases}$$





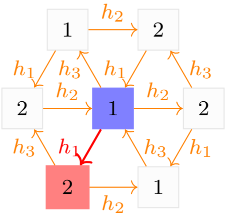
$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: Adding rule

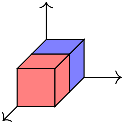
1. 2-atoms state $|\boxed{1}\boxed{2}\rangle$, with $h(\boxed{1}) = 0$, $h(\boxed{2}) = h_1$
2. Charge function for blue atoms:

$$\Psi_{\mathrm{K}}^{(1)}(z) = \psi_0^{(1)}(z) \cdot \varphi^{1\Leftarrow 1}(z - h(\boxed{1})) \cdot \varphi^{1\Leftarrow 2}(z - h(\boxed{2}))$$

$$= \frac{1}{\cancel{z}} \cdot \frac{\cancel{(z+h_3)}}{(z-h_3)} \cdot \frac{\cancel{z}(z+h_2-h_1)}{(z-2h_1)\cancel{(z-h_1-h_2)}}$$

(cancelation due to $h_1 + h_2 + h_3 = 0$)





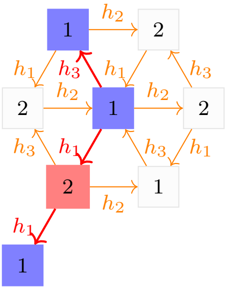
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-2

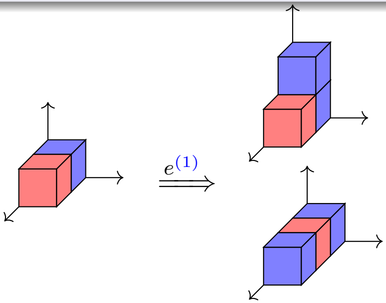
1. 2-atoms state $|\boxed{1}\boxed{2}\rangle$, with $h(\boxed{1})=0$, $h(\boxed{2})=h_1$
2. Charge function for blue atoms:

$$\Psi^{(1)}_{\mathrm{K}}(z)=\psi^{(1)}_0(z)\cdot\varphi^{1\Leftarrow 1}(z-h(\boxed{1}))\cdot\varphi^{1\Leftarrow 2}(z-h(\boxed{2}\ ))$$

$$=\frac{1}{\cancel{z}}\cdot\frac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\frac{\cancel{z}(z+h_2-h_1)}{(z-2h_1)\cancel{(z-h_1-h_2)}}$$

(cancelation due to $h_1+h_2+h_3=0$)





$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: Adding rule

1. 2-atoms state $|\boxed{1}\boxed{2}\rangle$, with $h(\boxed{1})=0$, $h(\boxed{2})=h_1$
2. Charge function for blue atoms:

$$\Psi^{(1)}_{\mathrm{K}}(z)=\psi_0^{(1)}(z)\cdot\varphi^{1\Leftarrow1}(z-h(\boxed{1}))\cdot\varphi^{1\Leftarrow2}(z-h(\boxed{2}))$$

$$=\frac{1}{\cancel{z}}\cdot\frac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\frac{\cancel{z}(z+h_2-h_1)}{(z-2h_1)\cancel{(z-h_1-h_2)}}$$

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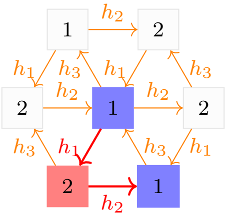
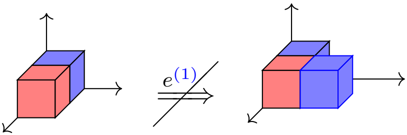


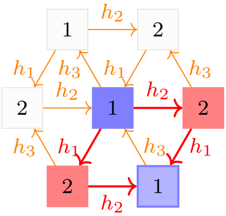


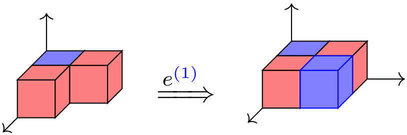
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: Adding rule

1. 3-atoms state $|\boxed{1}\boxed{2}_1\boxed{2}_2\rangle$, with $h(\boxed{1}) = 0$, $h(\boxed{2}_1) = h_1$ $h(\boxed{2}_2) = h_2$
2. Charge function for blue atoms:

$$\Psi_{\mathrm{K}}^{(1)}(z) = \psi_0^{(1)}(z)\cdot\varphi^{1\Leftarrow1}(z-h(\boxed{1}))\cdot\varphi^{1\Leftarrow2}(z-h(\boxed{2}_1))\cdot\varphi^{1\Leftarrow2}(z-h(\boxed{2}_2))$$
$$= \frac{1}{\cancel{z}}\cdot\frac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\frac{\cancel{z}(z+h_2-h_1)}{(z-2h_1)\cancel{(z-h_1-h_2)}}\cdot\frac{z(z+h_1-h_2)}{(z-2h_2)(z-h_1-h_2)}$$

(cancelation due to $h_1+h_2+h_3=0$)





$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: Adding rule

1. 3-atoms state $|\boxed{1}\boxed{2}_1\boxed{2}_2\rangle$, with $h(\boxed{1}) = 0$, $h(\boxed{2}_1) = h_1$ $h(\boxed{2}_2) = h_2$
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$$= \frac{1}{\cancel{z}} \cdot \frac{\cancel{(z+h_3)}}{(z-h_3)} \cdot \frac{\cancel{z}(z+h_2-h_1)}{(z-2h_1)\cancel{(z-h_1-h_2)}} \cdot \frac{z(z+h_1-h_2)}{(z-2h_2)(z-h_1-h_2)}$$

(cancelation due to $h_1 + h_2 + h_3 = 0$)

Poles of $\Psi_{\mathrm{K}}^{(a)}(z)$ encode the positions of $\boxed{a} \in \mathrm{Add}(\mathrm{K})$ and $\mathrm{Rem}(\mathrm{K})$

Poles are always pushed to the surface of crystal !

Only necessary poles are present. No redundant pole.

"Adding rule" is automatically implemented.

Quadratic relations in BPS algebra *L-Yamazaki '20*

$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\ , \\
\Big[e^{(a)}(z), f^{(b)}(w)\Big\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\ , \\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\ , \\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,e^{(a)}(z)\ , \\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\ , \\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\ .
\end{aligned}
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Quadratic relations in BPS algebra *L-Yamazaki '20*

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\end{aligned}
$$

Read off range of mode-expansions of $(e^{(a)}(z), \psi^{(a)}(z), f^{(a)}(z))$ from algebra's action on crystals:

- e/f :

$$
e^{(a)}(z) = \sum_{j=0}^{\infty} \frac{e_j^{(a)}}{z^{j+1}} \quad \text{and} \quad f^{(a)}(z) = \sum_{j=0}^{\infty} \frac{f_j^{(a)}}{z^{j+1}}
$$

Quadratic relations in BPS algebra *L-Yamazaki '20*

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- Using $\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}}\frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)}$

w/o compact 4-cycle $\Rightarrow$ non-chiral quiver $\Rightarrow$ homogenous $\varphi^{a\Leftarrow b}(u)$

w compact 4-cycle $\Rightarrow$ chiral quiver $\Rightarrow$ inhomogenous $\varphi^{a\Leftarrow b}(u)$

$$\Longrightarrow \quad \psi^{(a)}(z) = \begin{cases} \sum_{j=0}^{\infty}\frac{\psi_j^{(a)}}{z^{j+1}} & \text{(w/o compact 4-cycle)} \\ \sum_{j=-\infty}^{\infty}\frac{\psi_j^{(a)}}{z^{j+1}} & \text{(w/ compact 4-cycle)} \end{cases}$$

Quadratic relations in BPS algebra *L-Yamazaki '20*

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\end{aligned}
$$

Plug in mode expansions to algebraic relations and take singular terms:

- For all quivers

$$
\Big[\psi^{(a)}_n\,,\,\psi^{(b)}_m\Big] = 0 \quad \text{and} \quad \Big[e^{(a)}_n\,,\,f^{(b)}_m\Big\} = \delta^{a,b}\,\psi^{(a)}_{n+m}
$$

Quadratic relations in BPS algebra *L-Yamazaki '20*

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- ψ-e, ψ-f, e-e and f-f relations depend on quiver:

$$\mathbb{C}^3: \quad \begin{aligned} h_1h_2h_3\{\psi_j, e_k\} = &\,([\psi_{j+3}, e_k] - 3[\psi_{j+2}, e_{k+1}] + 3[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}]) \\ &+ (h_1h_2 + h_2h_3 + h_3h_1)\,([\psi_{j+1}, e_k] - [\psi_j, e_{k+1}]) \end{aligned}$$

Quadratic relations in BPS algebra *L-Yamazaki '20*

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confirmed experimentally from $\mathcal{N}=4$ quiver QM *Galakhov-Yamazaki '20*

Higher order relations in BPS algebra *L-Yamazaki '20*

- How do we know we have found all the higher order relations?
- BPS algebra should capture entire $\mathcal{H}_{\text{BPS}}$:

$$\text{vacuum character of BPS algebra} = Z_{\text{BPS}} = Z_{\text{crystal}}$$

Higher order relations in BPS algebra *L-Yamazaki '20*

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- Demanding this can determine all higher order relations

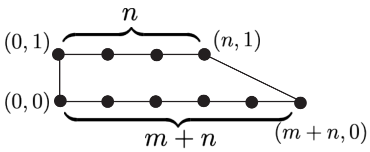
  Reproduce Serre relations for affine Yangian of $\mathfrak{gl}_{n|m}$

$$\text{e.g. } \mathbb{C}^3 : \qquad \begin{aligned} \text{Sym}_{(z_1,z_2,z_3)}(z_2-z_3)\, e(z_1)\, e(z_2)\, e(z_3) \sim 0 \\ \text{Sym}_{(z_1,z_2,z_3)}(z_2-z_3)\, f(z_1)\, f(z_2)\, f(z_3) \sim 0 \end{aligned}$$

- Open problem: find all higher order relations for general quiver Yangians

Special cases

- Toric CY_3: $xy = z^m w^n$



$$\text{quiver Yangian} \quad = \quad \text{affine Yangian of } \mathfrak{gl}_{m|n}$$

Ueda '19

Generalize to trigonometric and elliptic version

Galakhov-L-Yamazaki '21

- bond factor

$$\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}} \frac{\prod_{I\in\{a\to b\}} \zeta(u+h_I)}{\prod_{J\in\{b\to a\}} \zeta(u-h_J)}$$

- rational $\longrightarrow$ trigonometric $\longrightarrow$ elliptic

$$\zeta(u) \equiv \begin{cases} u & \text{(rational)} \quad \Longrightarrow \quad \text{quiver Yangians} \\ \sim \sinh \beta u & \text{(trig.)} \quad \Longrightarrow \quad \text{toroidal quiver algebras} \\ \sim \theta_q(u) & \text{(elliptic)} \quad \Longrightarrow \quad \text{elliptic quiver algebras} \end{cases}$$

- Bootstrap from crystal representation before central extension
- Turn on central extension and fix by consistency
- Confirm from gauge theory (2D $(2,2)$ and 3D $\mathcal{N}=2$ theory)

Outline

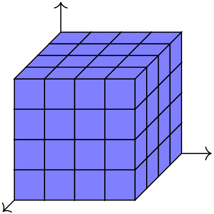
So far: canonical crystal

- So far: the crystal starts with a single atom and grows infinitely.
- This is realized by

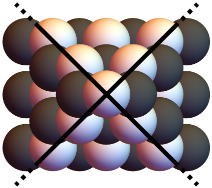
$$\text{ground state charge function:} \quad \psi_0^{(a)}(z) = \left(\frac{1}{z}\right)^{\delta_{a,\mathfrak{o}}}$$

- For example: the shape of the infinite crystals are

$\mathbb{C}^3$



resolved conifold



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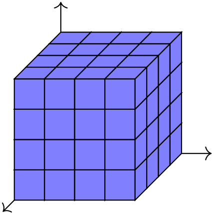

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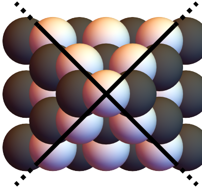
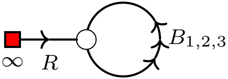


- Translate to framing:

$Q_0 =$



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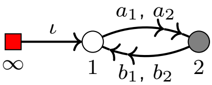
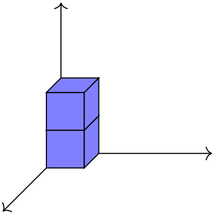


- Corresponds to counting of closed BPS invariants in the NCDT chamber.

From canonical crystal to other crystals

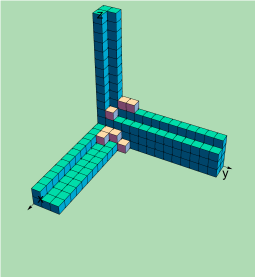
- How to describe open BPS states and/or other chambers?
- open BPS states (additional D2 ending on D4 wrapping Lag cycles in CY_3)

  Eg ($\mathbb{C}^3$):



Closed: plane partitions

$\Longrightarrow$



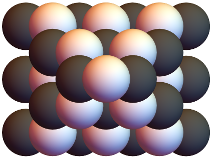
Open: plane partition with non-trivial asympotics

(labeled by three Young diagrams)

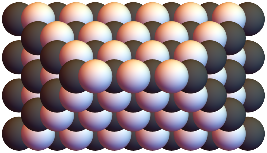
From canonical crystal to other crystals

- How to describe open BPS states and/or other chambers?
- Wall-crossing to other chambers

  Eg (resolved conifold):



$\Longrightarrow$



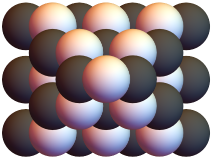
Non-commutative DT chamber

chamber with three initial atoms

From canonical crystal to other crystals

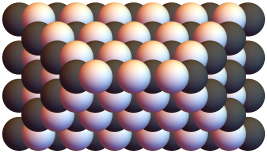
- How to describe open BPS states and/or other chambers?
- Wall-crossing to other chambers

  Eg (resolved conifold):



Non-commutative DT chamber

$\Longrightarrow$



chamber with three initial atoms

- How to describe crystals with different shapes (using canonical crystal)?

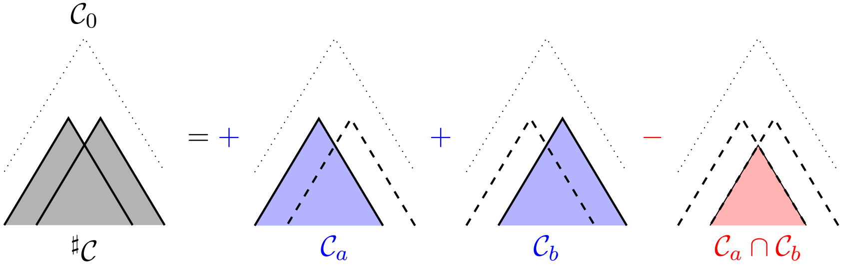
For a given (Q, W)

1. Subcrystal of canonical crystal (different boundary with same internal structure).
2. Can decompose arbitrary subcrystals as superposition of positive/negative canonical crystals

Decomposing subcrystal $^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

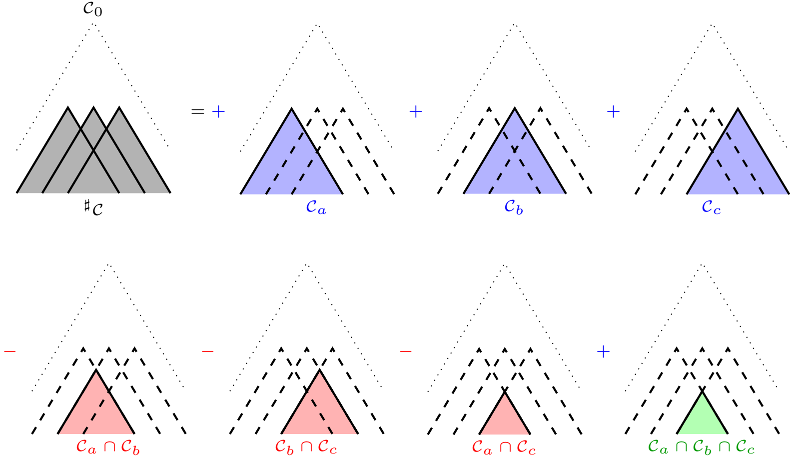
- step-1: near the starting point of $^\sharp\mathcal{C}$, determine the positions of positive $\mathcal{C}_0$

- step-2: determine the overlaps of positive $\mathcal{C}_0$
  $\Longrightarrow$ add negative $\mathcal{C}_0$ to cancel the overlaps



Decomposing subcrystal $^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

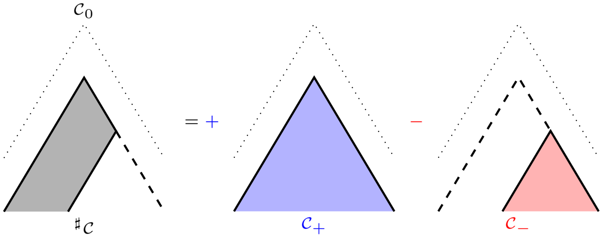
- step-3: determine the overlaps of negative $\mathcal{C}_0$
  $\Longrightarrow$ add positive $\mathcal{C}_0$ to cancel overlaps of negative $\mathcal{C}_0$



- step-4: continue until $^\sharp\mathcal{C}$ is reproduced (inclusion-exclusion principle)

Decomposing subcrystal ${}^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

- (optional) final step: truncate by adding negative $\mathcal{C}_0$



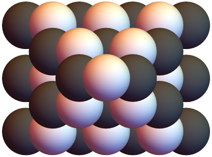
Can use this to describe 2D slices of crystal
(e.g. $\mathbb{C}^3$: Fock module v.s. MacMahon module)

Can generate very general simply-connected subcrystals this way.

For more examples, see *Galakhov-L-Yamazaki '21*

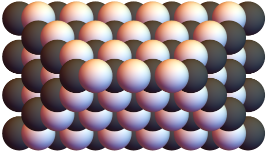
Decomposing subcrystal $^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

- Eg: different chambers for resolved conifold:



Non-commutative DT chamber

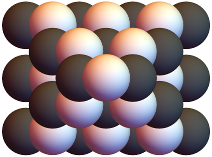
$\Longrightarrow$



chamber with three initial atoms

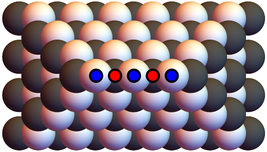
Decomposing subcrystal $^{\sharp}\mathcal{C}$ into positive/negative $\mathcal{C}_0$

- Eg: different chambers for resolved conifold



Non-commutative DT chamber

$\Longrightarrow$



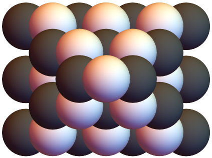
chamber with three initial atoms

3 positive canonical crystals

2 negative canonical crystals

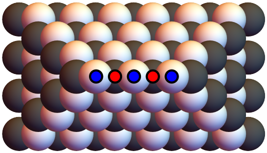
Decomposing subcrystal $^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

- Eg: different chambers for resolved conifold



Non-commutative DT chamber

$\Longrightarrow$



chamber with three initial atoms

Subcrystal $^\sharp\mathcal{C}$ describe non-vacuum representations

vacuum representation

$$\psi_0^{(a)}(z) = \left(\frac{1}{z}\right)^{\delta_{a,1}}$$

non-vacuum representation

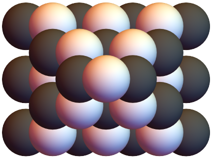
$$^\sharp\psi_0^{(a)}(z) = \frac{\prod_{\beta=1}^{\mathtt{s}_-^{(a)}}(z - {z_-}_\beta^{(a)})}{\prod_{\alpha=1}^{\mathtt{s}_+^{(a)}}(z - {z_+}_\alpha^{(a)})}$$

pole ${z_+}^{(a)}$ = weight of first atom of positive $\mathcal{C}_0$

zero ${z_-}^{(a)}$ = weight of first atom of negative $\mathcal{C}_0$

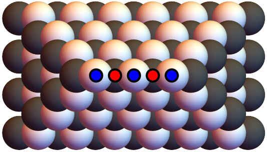
Decomposing subcrystal ${}^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

- Eg: different chambers for resolved conifold



Non-commutative DT chamber

$\Longrightarrow$



chamber with three initial atoms

Need more general mode expansions of $\psi^{(a)}(z)$
(define "shifted" quiver Yangians)

$$\psi^{(a)}(z) = \sum_{j}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1}}$$

unshifted quiver Yangian

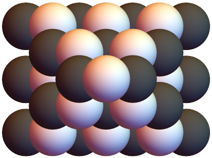
$$\psi^{(a)}(z) = \sum_{j}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1+\mathtt{s}^{(a)}}}$$

quiver Yangian with shift $\mathtt{s}^{(a)} \equiv \mathtt{s}_+^{(a)} - \mathtt{s}_-^{(a)}$

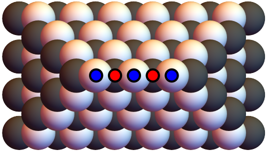
Galakhov-L-Yamazaki '21

From subcrystal to framed quiver

- Eg: different chambers for resolved conifold



$\Longrightarrow$

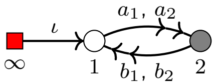


Non-commutative DT chamber

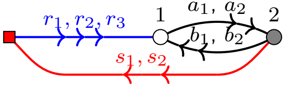
chamber with three initial atoms

Different representations $\Longleftrightarrow$ diffrent framings

$Q_0 =$



$Q_3^{(i)} =$



$$W = \mathrm{Tr}\left[b_2 a_2 b_1 a_1 - b_2 a_1 b_1 a_2\right].$$

$$W_3^{(i)} = \mathrm{Tr}[b_2 a_2 b_1 a_1 - b_2 a_1 b_1 a_2 + \sum_{i=1}^{2} s_i \left(a_2 r_i - a_1 r_{i+1}\right)].$$

Outline

Summary: BPS algebra for IIA string on general toric CY_3

1. Bootstrap construction

$\frac{1}{2}$-BPS sector: $\mathcal{N}=4$ quiver quantum mechanics (Q, W)

$$\Downarrow \quad \text{define}$$

{ BPS states } = { 3d colored crystals }

$$\text{act} \quad \Uparrow \Downarrow \quad \text{bootstrap}$$

BPS algebra: quiver Yangian $Y(Q, W)$

$$\begin{aligned}
\psi^{(a)}(z)\, \psi^{(b)}(w) &= \psi^{(b)}(w)\, \psi^{(a)}(z) \,, \\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b} \frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w} \,, \\
\psi^{(a)}(z)\, e^{(b)}(w) &\simeq \varphi^{a \Leftarrow b}(z-w)\, e^{(b)}(w)\, \psi^{(a)}(z) \,, \\
e^{(a)}(z)\, e^{(b)}(w) &\sim (-1)^{|a||b|} \varphi^{a \Leftarrow b}(z-w)\, e^{(b)}(w)\, e^{(a)}(z) \,, \\
\psi^{(a)}(z)\, f^{(b)}(w) &\simeq \varphi^{a \Leftarrow b}(z-w)^{-1}\, f^{(b)}(w)\, \psi^{(a)}(z) \,, \\
f^{(a)}(z)\, f^{(b)}(w) &\sim (-1)^{|a||b|} \varphi^{a \Leftarrow b}(z-w)^{-1}\, f^{(b)}(w)\, f^{(a)}(z) \,.
\end{aligned}$$

Summary: BPS algebra for IIA string on general toric CY_3

1. Bootstrap construction

   $\frac{1}{2}$-BPS sector: $\mathcal{N}=4$ quiver quantum mechanics (Q,W)

   $\Downarrow$ define

   { BPS states } = { 3d colored crystals }

   act $\Uparrow$ $\Downarrow$ bootstrap

   BPS algebra: quiver Yangian $Y(Q,W)$

2. Generalized to trigonometric and elliptic versions

3. Subcrystals (or different framing) give non-vacuum representations
   (describing other chambers, open BPS invariants, and more)

Applications

Computing BPS partition functions

- Subcrystal representations: include different chambers, open BPS, and more.

$$Z_{\mathrm{subcrystal}} = Z_{\mathrm{BPS}}$$

- Counting poles of Ψ_{K} gives an efficient way to compute $Z_{\mathrm{subcrystal}}$.

mathematica program by B. Pioline

Deriving Bethe Ansatz Equation in Gauge/Bethe correspondence (for non-chiral quivers) *Nekrasov Shatashvili '09*

- spin-chain $\longrightarrow$ crystal-chain
- Express Lax operators, transfer matrices, off-shell Bethe vectors in terms of quiver Yangian generators
- BAE reproduces vacuum equation of the corresponding quiver gauge theory

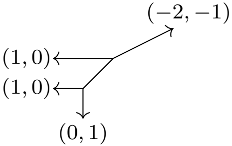
Galakhov-L-Yamazaki '22

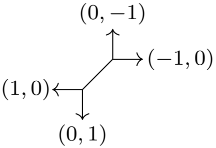
Future directions

- Dictionary to other formulations of BPS algebras
  - map to VOA?
  - map to Doubled CoHA?
- Relation to gluing constructions *L-Longhi '19, L '19*

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ Resolved conifold

e.g.





- Test the conjecture that quiver Yangins related by quiver mutation are isomorphic *L-Yamazaki '20*
- Characterize all the subcrystal representations in terms of D-branes
- Generalize to toric Calabi-Yau fourfolds or even beyond

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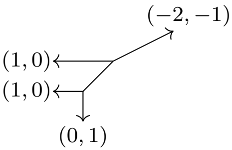

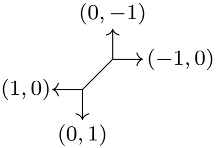

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Thank you for your attention !

Gauge/Bethe correspondence from quiver BPS algebras

Galakhov-L-Yamazaki '22