String Math 2022

Smoothing, scattering and a conjecture of Fukaya

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Calabi-Yau manifold

(X, ω, J) Kähler, $\dim_{\mathbb{C}} X = n$

$c_1(X) = 0$

$\Rightarrow$ (1) $\exists$ holom. volume form $\Omega \in \Omega^{n,0}(X)$

s.t. $\Omega \overline{\Omega} = \omega^n / n!$ (Yau)

(2) complex moduli $\mathcal{M}$ is smooth. (Bogomolov-Tian-Todorov)

$\mathcal{M} = \{$ complex structures on $X\} / \mathrm{Diff}(X)$

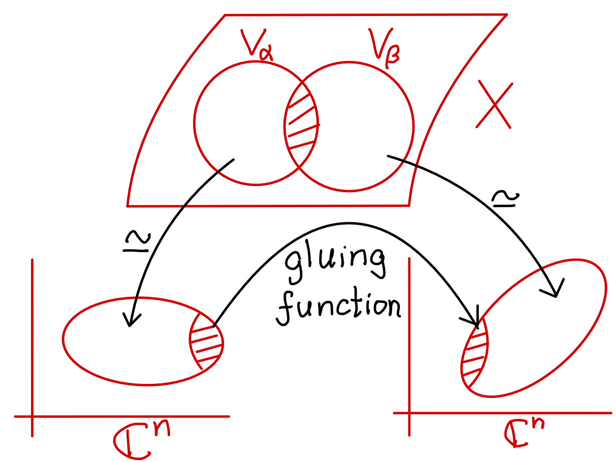
(BTT) $\mathcal{M}$ smooth

(i.e. infinitesimal deformations $H^1(X, T_X)$ are unobstructed)

1° Čech approach

$X = \bigcup_\alpha V_\alpha$

vary gluing functions



2° Differential forms approach

$$\bar{\partial} \rightsquigarrow \bar{\partial} + \varphi$$

$$\varphi \in \Omega^{0,1}(X, T_X) \subset (\Omega^{0,*}(X, T_X), \bar{\partial}, [\ \ ])$$

dgla (dg Lie algebra)

satisfying Maurer-Cartan equation (MC eqt)

$$\bar{\partial}\varphi + \frac{1}{2}[\varphi, \varphi] = 0 \quad (\Longleftrightarrow (\bar{\partial} + \varphi)^2 = 0)$$

Taylor expansion: $\varphi(t) = t\varphi_1 + t^2\varphi_2 + \cdots\cdots$

MC eqt $(\bar\partial + \varphi(t))^2 = 0$ becomes

$$\bar\partial \varphi_1 = 0 \xRightarrow{\text{infinitesimally}} \frac{d}{dt}\Big|_{t=0}[X_t] = [\varphi_1] \in H^1_{\bar\partial}(X, T_X)$$

$\bar\partial \varphi_2 = -\frac{1}{2}[\varphi_1, \varphi_1]$, and so on

CY $\Longrightarrow$ always solvable

$$\Omega^{0,j}(X,\Lambda^i T_X) \xrightarrow[\cong]{\lrcorner\Omega} \Omega^{n-i,j}(X)$$

CY

BV operator $\Delta \longleftarrow\!\!\sim \partial$

Tian-Todorov lemma:

$$[\varphi,\eta] = \Delta(\varphi\wedge\eta) - \Delta\varphi\wedge\eta \pm \varphi\wedge\Delta\eta$$

denote $PV^{i,j}(X) = \Omega^{0,j}(X,\Lambda^i T_X)$ (polyvector fields)

$(PV^{*,*}(X), \bar\partial, \wedge, [\ \ ], \Delta)$ dgBV algebra

Bogomolov-Tian-Todorov theorem

$$CY \Longrightarrow \exists \text{ MC solution}$$

Proof:

$$[\varphi_1] \in H^1(X, T_X) \xrightarrow[\sim]{\lrcorner\Omega} H^{n-1,1}_{\bar\partial}(X) \overset{(\Delta_{\bar\partial} = \Delta_\partial)}{\underset{\text{Hodge-deRham degeneration}}{=\!=\!=}} H^{n-1,1}_{\partial}(X)$$

choose φ_1 w/ $\Delta \varphi_1 = 0$

$$[\varphi_1, \varphi_1] \overset{\text{TT lemma}}{=\!=\!=} \Delta \varphi_1^2 = -\bar\partial \varphi_2, \quad \exists\, \varphi_2 \in \Omega^{0,1}(X, T_X)$$

Inductively, $\exists\, \varphi(t) = t\varphi_1 + t^2\varphi_2 + t^3\varphi_3 + \cdots\cdots \in \Omega^{0,1}(X, T_X)[[t]]$

solving MC eqt $\quad (\bar\partial + \varphi(t))^2 = 0.$

More generally, we can solve

$$(\bar\partial + z\Delta + \varphi)^2 = 0 \text{ in } \Omega^{0,*}(X, \Lambda^* T_X) = PV^{*,*}(X)$$

$$\xrightarrow{\text{Barannikov}} \frac{\infty}{2}\text{-VHS on } \mathcal{M}$$

Rk: Differential form approach gives a direct proof of unobstructedness, which is not as transparent in Čech approach.

<u>Mirror Symmetry conjecture</u>: CY n-folds $\check{X}$, X

$$\mathrm{Sympl}(\check{X}, \check{\omega}) \cong \mathrm{Complex}(X, \Omega)$$

<u>Kontsevich HMS conj.</u> $\mathcal{F}uk(\check{X}, \check{\omega}) \cong D^b(\mathrm{Coh}(X, \Omega))$

<u>Strominger - Yau - Zaslow conjecture</u>:

$$\begin{tikzcd}
\check{X} \arrow[d, "\check{T}"'] & & X \arrow[d, "T"] \\
\check{B} \arrow[rr, leftrightarrow, "\text{Legendre}", "\text{dual}"'] & & B
\end{tikzcd}$$

dual torus

special Lagrangian fibrations

near

large volume/complex str. limits

(LVL / LCSL)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{A} & \text{B} \\
\text{General } (\check{X}, \check{\omega}) \arrow[r, dashed, "\text{SYZ}", "?"'] \arrow[d, "k\check{\omega} \,\to\, \infty\check{\omega}"] & (X, \Omega) \\
\text{LVL} \arrow[r, "\text{dual}", "\text{T-fiber}"'] & \text{LCSL} \arrow[u, dashed, "\text{smoothing}"']
\end{tikzcd}
$$

quantum correction $\to$ zero

$e^{-\int_{D^2} k\check{\omega}} \to 0$

smoothing (deformation "from infinity")

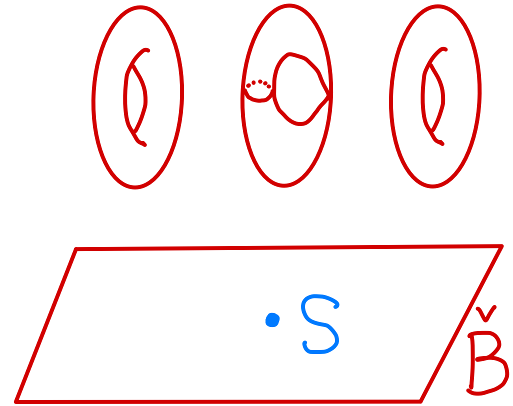
(singular) semi-flat

Start

$$\begin{array}{c} (\check{X}, \check{\omega}) \\ \downarrow \check{T} \\ \check{B} \end{array}$$

Lagr. fibration

$\mathbb{Z}$-affine manifold w/ singularities S



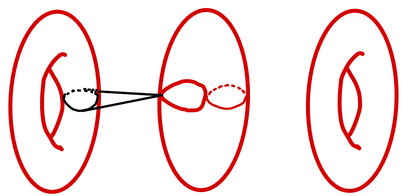
$$\check{X}\big|_{\check{B}^\circ} = \frac{T^*\check{B}^\circ}{T^*_{\mathbb{Z}}\check{B}^\circ} \quad \text{over} \quad \check{B}^\circ = \check{B} \setminus S \,,$$

$$\xrightarrow[\text{fibration}]{\text{dual toric}} \; X^{sf} = \frac{T\check{B}^\circ}{T_{\mathbb{Z}}\check{B}^\circ}$$

semi-flat complex manifold

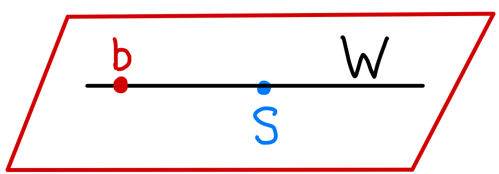
Fukaya proposal

1° Singular fibers $\overset{\text{give}}{\rightsquigarrow}$

$D^2 \xrightarrow[\text{holo.}]{u}$  $\check{X}$

$\cup$

$\partial D^2 \longrightarrow L_b = \mu^{-1}(b)$ $\qquad \check{X} \xrightarrow{\mu} \check{B}$



$\check{B}$

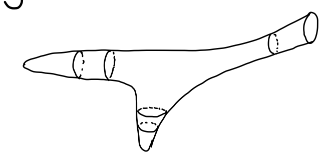
$\rightsquigarrow$ <u>Wall</u> $\quad W = \{b\} \overset{\text{codim } 1}{\subset} B$

2° Gluing holomorphic disks

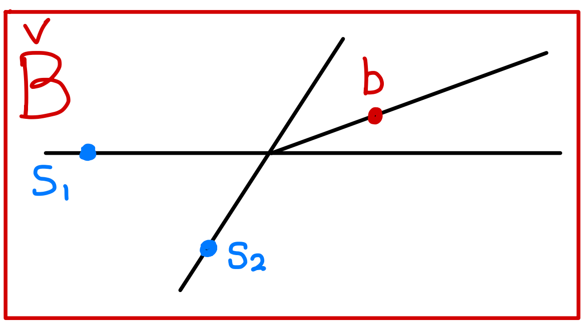
$\updownarrow$

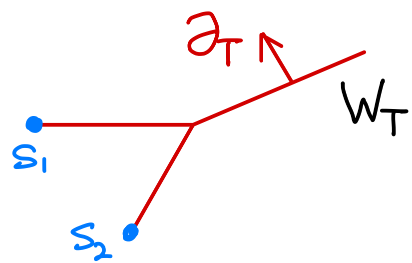
scattering diagram of walls

glued holom. disk



$\sim$ Gradient flow tree (tropical) T in $\check{B}$ (straight lines in B)





f_T = generating series of corresp. disk count.

such data should give X^{sf} to X.

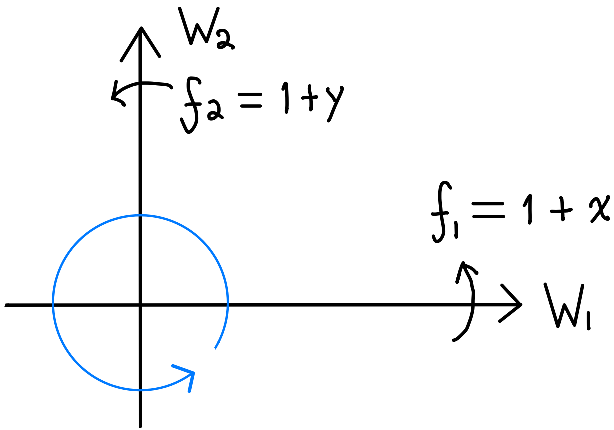
Kontsevich - Soibelman ($n = 2$)
Gross - Siebert ($\forall n$) reconstruction.

Wall structure $\sim$ consistent scattering diagram $\sim$ smoothing of LCSL to X
(holom. disk count in $\check{X}$) (combinatorial)

$S \subset \check{B} \rightsquigarrow$ initial walls

gluing disks $\rightsquigarrow$ new (scattered) walls
(controlled combinatorially by monodromy.)

Scattering diagram $\rightsquigarrow \xrightarrow{\text{add new walls}}$ consistency

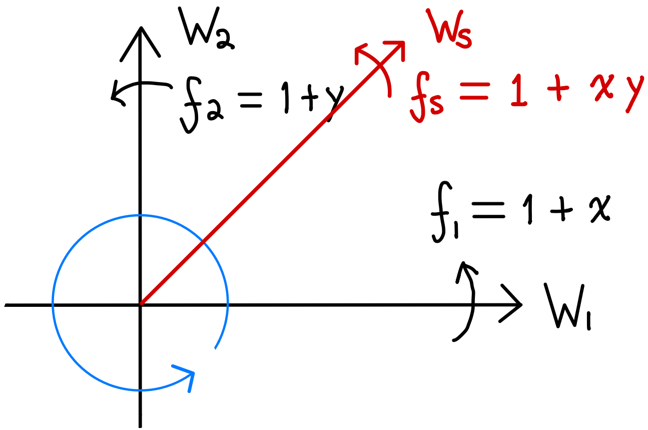


$$\Theta_i = \exp(\log f_i \otimes \partial_i) \in \mathrm{Aut}(\mathbb{C}^{\times 2})$$

$$\Theta_2^{-1}\Theta_1^{-1}\Theta_2 \quad \Theta_1 \neq \mathrm{id}$$

gluing consistency (need new (W_s, f_s))

Scattering diagram $\rightsquigarrow$ add new walls $\longrightarrow$ consistency



$$\Theta_i = \exp(\log f_i \otimes \partial_i) \in \mathrm{Aut}(\mathbb{C}^{\times 2})$$

$$\Theta_2^{-1}\Theta_1^{-1}\Theta_2\Theta_s\Theta_1 = \mathrm{id}$$

consistency condition !

Gross-Siebert construction (B-side)

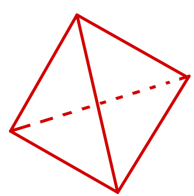
Čech approach to construct smoothing of LCSL/toric degenerations

Eg $\quad X_t = \{ t \sum_{i=0}^{3} x_i^4 + \prod_{i=0}^{3} x_i = 0 \} \subset \mathbb{P}^3$

$t \to 0 \downarrow$

$$X_0 = \bigcup_{i=0}^{3} \mathbb{P}^2, \quad X_0^{reg} = \bigcup^{4} \mathbb{C}^{\times 2} = \underbrace{\bigcup^{4} T(S^1)^2}_{\text{LCSL}}$$

moment maps $\bigcup_{i=0}^{3} \mu_i \downarrow$

$B =$ 

$$
\begin{tikzcd}[column sep=huge]
 & & X_0 \arrow[r, "\text{scattering diagram}", "\text{Čech approach}"'] & X_t\text{'s} \\
(\check{B}, \check{\mathscr{P}}, \check{\varphi}) \arrow[r, leftrightarrow, "\text{Legendre}", "\text{dual}"'] & (B, \mathscr{P}, \varphi) \arrow[ur, "\text{toric}"] & &
\end{tikzcd}
$$

mirror family.

$\mathscr{P}$: polyhedral decomposition

φ: convex piecewise affine function
(~ample line bundle on X_0)

Differential form approach ?

$$\{\text{complex str.}\} = \{\text{MC sol}^{\underline{n}}\} \subset \text{dgla}$$

(up to $\cong$)

1° MC solution

= consistent scattering diagram.

2° LCSL $X_0 \rightsquigarrow X_t$ via

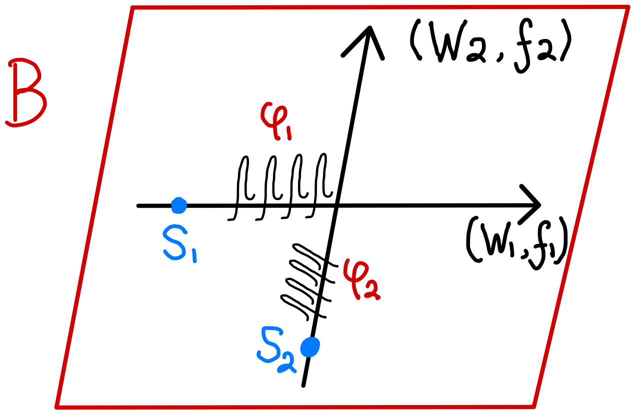
deformation via dgla.

<u>"δ-function" type forms</u> from walls (W_T, ∂_T, f_T)

$$\varphi := \sum_T \delta_{W_T} \otimes f_T \otimes \partial_T \in \Omega^{0,1}(X^{sf}, T^{1,0})$$

$\rightsquigarrow$ deform X^{sf} to X !

1° Maurer-Cartan solution $\Longrightarrow$ consistent scattering diagram



$\forall$ wall $(W_i, f_i) \rightsquigarrow$

$$\varphi_i = \delta_{i,\hbar} \otimes \log f_i \otimes \partial_i$$
$$\in \Omega^{0,1}(X^{sf}, T_X^{1,0})$$

$\left(X^{sf} = \frac{T^*B}{T^*_{\mathbb{Z}}B}\right)$ ($\delta_{i,\hbar}$: smoothing of δ-function)

(only consider diff. form $\sim \delta_{\text{polyhedron}}$.)

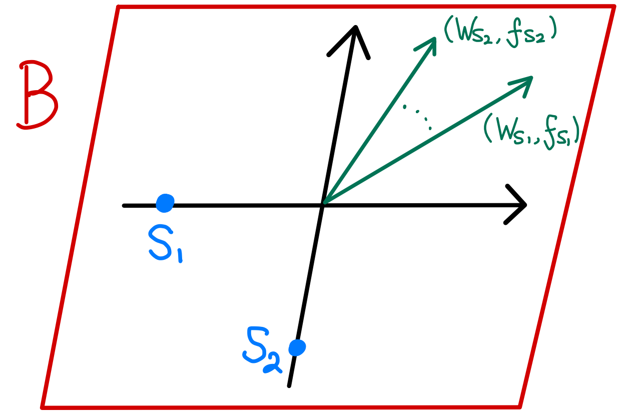
Scattering occurs on B^{reg}. Local problem.

Each φ_i satisfies MC eqt., but $\varphi_1 + \varphi_2$ does not.

<u>Theorem 1</u> (Chan-L.-Ma)

$$\varphi_1 + \varphi_2 + \underbrace{\sum_{(w_{s_j}, f_{s_j})} \delta_{w_{s_j},\hbar} \otimes \log f_{s_j} \otimes \partial_{w_{s_j}}}_{\varphi_s}$$

solves MC eqt.



(ref: Chan-L.-Ma, Scattering diagrams from asymptotics analysis on MC eqt. (JEMS))

Pf: • MC eqt is solved via Kurinishi summation over trees formula.

• Asymptotic analysis on solutions.

2° Maurer-Cartan solution $\Longrightarrow$ Smoothing

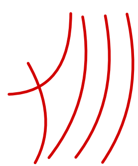
Recall: (classical case) Start $(X, \bar\partial)$

Step 1. Consider $X \times \mathbb{C}_t$ w/ $(\bar\partial)^2 = 0$ at $t = 0$.

Step 2. Solve MC eqt. $(\bar\partial + \varphi(t))^2 = 0$ $\forall t$ via BTT

[Smoothing]

Step 1 is highly nontrivial for smoothing.

$X \times \mathbb{C} \rightsquigarrow$  $\mathcal{X}_t$ w/ relative (pre-) dgBV $(PV^{*,*}, \bar\partial, \wedge, [\ ], \Delta)$

(classical)

except $(\bar\partial)^2 = 0$ <u>only</u> at $t=0$

Step 2. pre-dgBV $\xRightarrow{BTT}$ solve MC eqt.

$\rightsquigarrow$ smoothing

Theorem 2 (Chan-L.-Ma)

(ref: Chan-L.-Ma, Geometry of MC eqt. near degen. CY varieties, (JDG))

(1) Construct pre-dgla $PV^{*,*}$ by gluing $PV^{*,*}_{V_\alpha/\mathbb{C}}$'s

(2) Solve MC eqt via BTT approach

$\rightsquigarrow$ smoothing of LCSL.

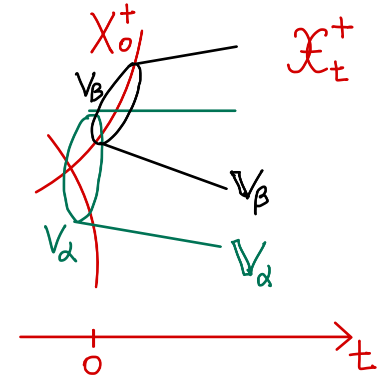
Cor. $\xrightarrow[\text{of Barannikov}]{\text{techniques}}$ $\frac{\infty}{2}$-VHS near LCSL.

Rk: This method is quite flexible.

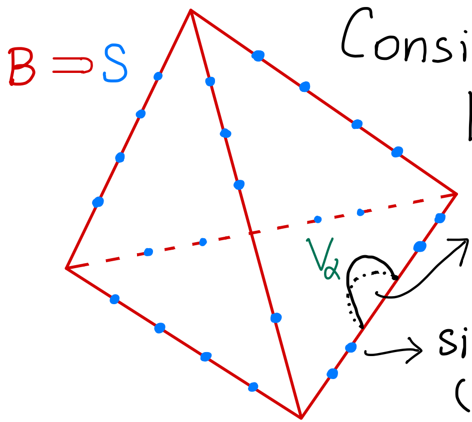
Can smooth other singular varieties besides LCSL. (Felten-Filip-Ruddat).

Pf of Thm 2: [Step 1]

pre-dgBV $PV^{*,*}$ is constructed via (non-holo) gluing of local models



To include singular set $S \subset B$



Consider

LCSL of K3 surface $X_0 = \bigcup^4 \mathbb{P}^2$

local model $\mathbb{V}_\alpha \rightsquigarrow PV^{*,*}_{\mathbb{V}_\alpha/\mathbb{C}}$ dg BV alg.

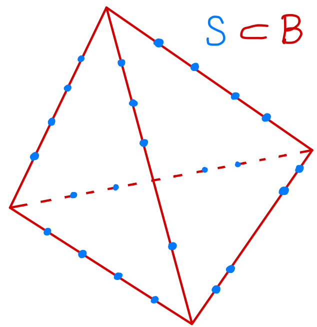
$\to$ singularity of log structure $\quad \mathbb{V} = \{uv = t f_\rho\}$

(zeros of $f_\rho \in H^0(\mathbb{P}^1, \mathcal{O}(4))$)

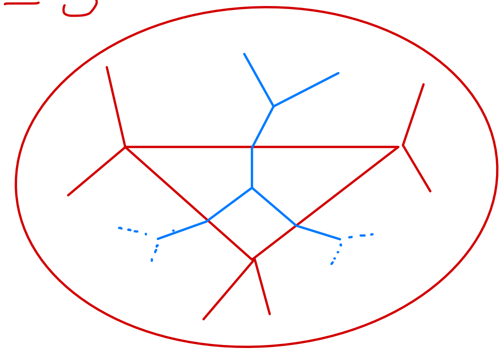
Gluing of $PV^{*,*}_{\mathbb{V}_\alpha/\mathbb{C}}$ is not easy!

$$\begin{tikzcd} X_0 = \bigcup_i X_{\sigma_i} \arrow[d, "\mu = \bigcup_i \mu_i"] \\ B = \bigcup_i \sigma_i \end{tikzcd}$$

$n=2$



$n=3$



via contracting amoebas to graphs.

- $\forall$ coherent sheaf $\mathcal{F} / X_0 \rightsquigarrow R^{>0}\mu_* \mathcal{F} = 0$
  ($\because$ contracting Steins domains)

$$\begin{tikzcd} X_0 \supset \underset{\text{Stein}}{V_\alpha = \mu^{-1}(\mu(V_\alpha))} \arrow[d, "\mu"'] \arrow[rr, rightsquigarrow, "\exists!\ \text{local model of smoothing}"'] & & V_\alpha \subset \mathbb{V}_\alpha \arrow[d] \\ B \supset \mu(V_\alpha) & & 0 \in \mathbb{C}_t \end{tikzcd}$$

$$PV_\alpha^{*,*} = \Omega^*\Big(\pi(V_\alpha), \underbrace{\mu_*\big(\Lambda^\circ \Theta_{V_\alpha/\mathbb{C}_q}(\log)\big)}_{\text{sheaf of Lie alg. over } \pi(V_\alpha)}\Big)$$

$$\text{glue} \Big\downarrow$$

$PV^{*,*}$ over B

Rk: Choice of resolution is quite flexible, eg Thom-Whitney resol$^{\underline{n}}$, this particular one can see the link to tropical geometry. (namely Thm 3).

[Step 2]

Maurer-Cartan equation :

tangent bundle

1 - form

$$(\bar\partial + \varphi)^2 = 0 \quad \text{for} \quad \varphi \in PV_\alpha^{-1,1}$$

$$\oplus$$

$$(\bar\partial + \varphi)(e^f\Omega) = 0 \quad \text{for} \quad f \in PV_\alpha^{0,0}$$

$$\Updownarrow$$

$$(\bar\partial + z\Delta)\Phi + \frac{1}{2}[\Phi, \Phi] = 0$$

where $\Phi = \varphi + zf$

Solved by BTT method $\Longrightarrow$ Thm. 2.

Theorem 3 (Chan-L.-Ma)

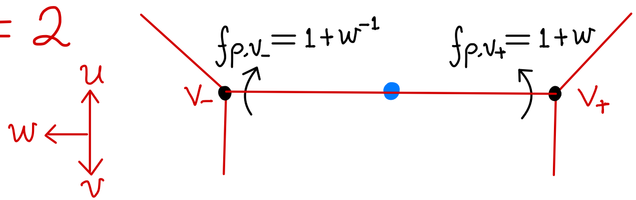
asym. expansion of Φ

$\rightsquigarrow$ global consistent scattering diagram

(ref: Chan-L.-Ma, Smoothing, scattering and conjecture of Fukaya, arXiv 2205.09926)

Remark: $X_0 = \bigcup_i X_{\sigma_i}$ vs $X^{sf} = \dfrac{T^*B^\circ}{T^*_{\mathbb{Z}}B^\circ}$

$n = 2$



$X_0 = \{uv = 0\} \times \mathbb{C}^\times_w$

$\downarrow$

$\mathcal{X}_t = \{uv = t(1+w)\}$

w/ monodromy on coordi. u, v, w

$\xrightarrow{\text{in family}}$ $\mathcal{X}_t^{sf}$

$$(PV^{*,*}, \bar\partial)\big|_{B^{\circ} \setminus \text{codim2 strata}} \cong (PV^{*,*}_{sf}, \bar\partial + \varphi_{in})$$

φ_{in} concentrated on initial walls $\varphi_{in}(t=0) \neq 0$

MC sol$^{\underline{n}}$ φ $\Downarrow$ THM 2

THM 3

$$(PV^{*,*}, \bar\partial + \varphi)\big|_{B^{\circ} \setminus \text{codim2 strata}} \longleftrightarrow (PV^{*,*}_{sf}, \bar\partial + \varphi_{in} + \varphi_{s})$$

φ_s concentrated on codim 1 walls

$\Downarrow$ THM 1

consistent scattering diagram