Calculating Quantum Knot Homology from Homological Mirror Symmetry

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String Math 2022

UC Berkeley

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Finally, I will explain how we are able to explicitly compute quantum knot homology in the A-model.

This joint work with Aganagic and Rapčák.

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Outline

1. Background: link invariants and categorification
2. The A-model
3. Links as A-branes
4. An algebraic approach
5. An algorithm using braiding

Background: link invariants and categorification

Link invariants

- Link invariants can be numbers, polynomials, homology groups, etc.
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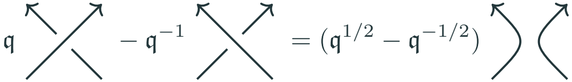
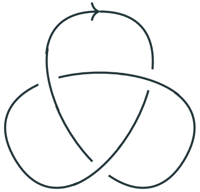
- However, no link invariant is known to distinguish all links.
- Example: the Jones polynomial, which is defined by the skein relation



Hopf link

$$J(\mathfrak{q}) = -\mathfrak{q}^{-1/2} - \mathfrak{q}^{-5/2}$$

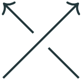
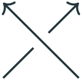
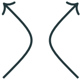
trefoil



$$J(\mathfrak{q}) = -\mathfrak{q}^{-4} + \mathfrak{q}^{-3} + \mathfrak{q}^{-1}$$

Other polynomial link invariants

The skein relation for the Jones polynomial can be generalized to

$\mathfrak{q}^{n/2}$  $- \mathfrak{q}^{-n/2}$  $= (\mathfrak{q}^{1/2} - \mathfrak{q}^{-1/2})$ 

to give a family of link invariants. Taking $n = 0$ gives the Alexander polynomial.

Chern-Simons theory

- Witten found the physical origin of these link invariants in 1989
- They come from Chern-Simons theory with gauge group $\mathfrak{g} = \mathfrak{su}_n$ with Wilson lines in the fundamental representation corresponding to the link
- The Alexander polynomial comes from $\mathfrak{g} = \mathfrak{gl}_{1|1}$
- By taking different gauge groups and different representations, one gets many more link invariants, called quantum group invariants

Categorification

The categorification program was pioneered by Crane and I. Frenkel.

It aims to lift

- integers to vector space,
- vector spaces to categories, and
- maps between vector spaces to functors between categories.

It should also “decatagorify” correctly.

Example of categorification

The Euler characteristic $\chi(M)$ of a Riemannian manifold M is categorified by de Rham cohomology on M.

From a physics perspective, as explained by Witten in 1982,

- $\chi(M)$ is the partition function of supersymmetric QM on M
- The de Rham operator d is the supercharge
- Given a Morse function h, one can replace d with $d_h = e^h d e^{-h}$
- h is a potential in the supersymmetric QM theory
- There is a chain complex spanned by pertubative ground states of H where the action of d_h produces instanton corrections

Categorifying link polynomials

- First done by Khovanov in 1998 for the Jones polynomial
- Khovanov homology assigns to a link a collection of bi-graded vector spaces

$$\mathcal{H}_K = \bigoplus_{i,j} \mathcal{H}_K^{i,j}$$

  such that the graded Euler characteristic coincides with the Jones polynomial:

$$J_K(\mathfrak{q}) = \sum_{i,j \in \mathbb{Z}} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{C}} \mathcal{H}_K^{i.j}$$

- Another example: knot Floer homology, which categorifies the Alexander polynomial

A physical approach to link categorification

- I will show how to categorify quantum group invariants for any simple Lie group with link components colored by a minuscule representation using the A-model
- We recover Khovanov homology when $\mathfrak{g} = \mathfrak{su}_2$
- Our approach also works for supergroups and we recover knot Floer homology for $\mathfrak{g} = \mathfrak{gl}_{1|1}$
- Better than previous approaches because
  - it gives a physical origin for link homology
  - it unifies link homologies associated with different groups

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- It is a $(1+1)$-dimensional version of the supersymmetric quantum mechanics mentioned earlier

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- We consider the derived Fukaya-Seidel category of Y with potential W
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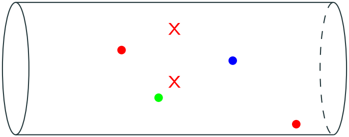
- One can think of Y as roughly the configuration space of colored points on a punctured infinite cylinder
- Example:



A-branes

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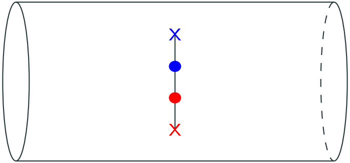
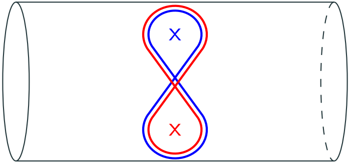
- They are the objects of the derived FS category
- They can be described as either
  - one-dimensional curves between a pair of punctures with ordered dots corresponding to simple roots of $\mathfrak{g}$
  - products of one-dimensional curves colored by simple roots on the punctured cylinder
- Examples:





Morphisms between branes

- Morphisms between branes are defined by Floer theory

$$hom^{*,*}_{\mathscr{D}_Y}(L_0, L_1) = \ker Q / \operatorname{im} Q$$

- This is a chain complex spanned by intersection points of the branes, with two gradings coming from W:
  - the $\mathfrak{q}$-grading $\vec{J}$
  - the Maslov grading M
- The differential Q comes from instantons, which have $(M, \vec{J}) = (1, 0)$

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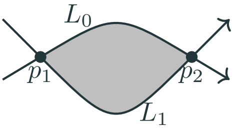
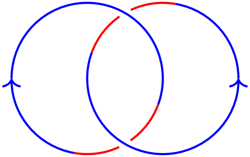
- Q can be computed, as in Heegaard-Floer theory, by counting holomorphic maps from a disk to Y



Links as A-branes

Translating a link to A-branes ($\mathfrak{su}_2$ example)

Start with a planar projection of a link and choose a bicoloring such that red segments always pass under blue segments:



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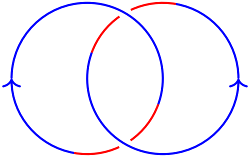
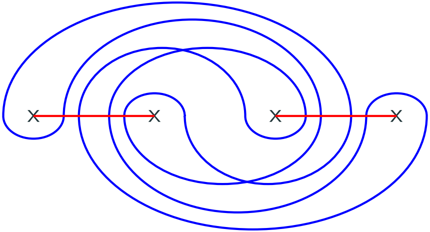


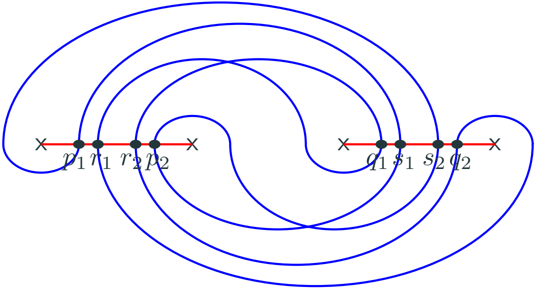
Replace red segments with interval branes $I_{\mathcal{U}}$ and blue segments with braided figure-eight branes $\mathscr{B}E_{\mathcal{U}}$:



Homological link invariants

The bigraded vector space $hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$ is a homological link invariant.

Here, the complex spanned by 8 points: $p_i q_j$, $r_i s_j$ for $i, j \in \{1, 2\}$.



One gets the differential by counting holomorphic disks with $(M, J_0) = (1, 0)$.

The Euler characteristic

One can recover the Jones polynomial from the intersection points:

$$\chi(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \sum_{\mathcal{P} \in \mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}} (-1)^{M(\mathcal{P})} \mathfrak{q}^{J_0(\mathcal{P})}$$

This is a theorem by Bigelow from the '90s.

An algebraic approach

Thimbles

- The category contains special branes known as thimbles
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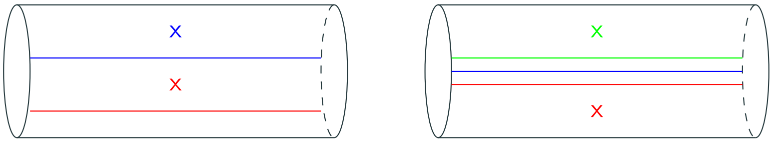

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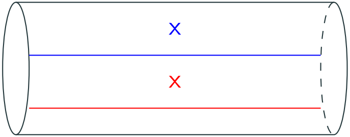
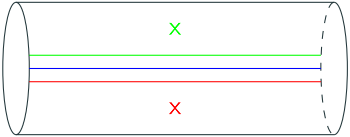




- Each thimbles passes through a critical point of W
- W also has critical points at ∞, leading an enlarged set of thimbles
- The algebra A is generated by morphisms between the thimbles
- A is closely related to the KLRW algebra and can be computed using mirror symmetry

Branes as thimbles

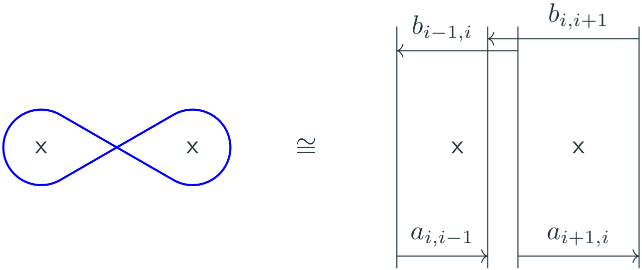
- The thimbles are projective modules of A
- Any brane in the derived FS category has a projective resolution in terms of thimbles
- This means we can write any brane as a chain complex of whose terms consist of direct sums of thimbles and whose maps are elements of A
- This represents a brane as the connected sum of thimbles glued according to the maps
- Using this projective resolution, we can describe the link homology purely algebraically using thimbles

Finding the projective resolution

1. Break the brane into a direct sum of thimbles
2. Add geometric maps connecting the thimbles corresponding to the geometry of the brane
3. Turn on additional maps (as allowed by degrees) so that $d^2 = 0$ in the algebra A

This last step is key and it is how we know where the disks enter into the chain complex

Example: $\mathfrak{su}_2$



Projective resolution:

$$E_i \cong T_i\{-1\} \xrightarrow{\begin{pmatrix} a_{i+1,i} \\ b_{i-1,i}\end{pmatrix}} \begin{matrix} T_{i+1}\{-1\} \\ \oplus \\ T_{i-1}\end{matrix} \xrightarrow{\begin{pmatrix} b_{i,i+1} & a_{i,i-1}\end{pmatrix}} T_i$$

Link invariants from projective resolutions

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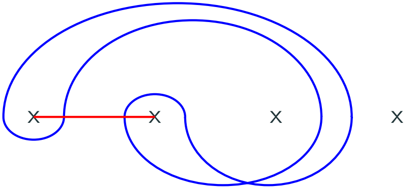
$$\implies \chi(E_i, I_i) = (-1)^2\mathfrak{q}^{-1} + (-1)^0\mathfrak{q}^0 = \mathfrak{q}^{-1} + 1$$

Example: $\mathfrak{su}_2$ Hopf link

To make things simpler to calculate, we can optionally cut a strand (as I will do here). This gives reduced quantum group invariants and reduced link homology.



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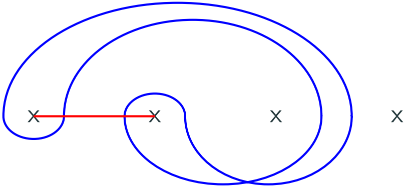


$$\mathscr{B}E_2 \cong T_3\{-3\} \xrightarrow{\begin{pmatrix} a_{4,3} \\ b_{2,3} \end{pmatrix}} \begin{matrix} T_4\{-3\} \\ \oplus \\ T_2\{-2\} \end{matrix} \xrightarrow{\begin{pmatrix} 0 & a_{4,2} \\ b_{1,4} & 0 \end{pmatrix}} \begin{matrix} T_4\{-2\} \\ \oplus \\ T_1 \end{matrix} \xrightarrow{\begin{pmatrix} b_{2,4} & a_{2,1} \end{pmatrix}} T_2$$

$$hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_2, I_2) \cong \mathbb{C}\{-2\}[2] \oplus \mathbb{C}$$

An algorithm using braiding

Braiding thimbles

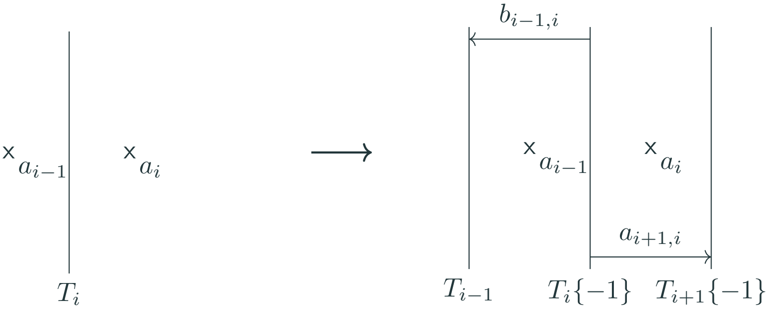
Exchanging punctures a_i and a_{i-1} clockwise sends T_i to

$$T_i\{-1\} \xrightarrow{\begin{pmatrix} a_{i+1,i} \\ b_{i-1,i} \end{pmatrix}} \begin{matrix} T_{i+1}\{-1\} \\ \oplus \\ T_{i-1} \end{matrix}$$

and leaves the other thimbles unchanged.

Braiding thimbles

Geometrically, this looks like



The algorithm

1. Start with projective resolution of figure-eight branes corresponding to particular Lie group and representation
2. Act with the braiding functor on projective resolution:
    2.1 Replace thimbles with result of braiding by a single generator
    2.2 Adjust maps attached to braided thimbles
    2.3 Simplify complex if possible
    2.4 Repeat for next generator
3. Use $d^2 = 0$ to fix non-geometric maps in complex
4. Intersect complex with interval branes to get homology

Summary and future directions

- We have a geometric formulation of link homology for minuscule reps of any simple Lie group
- We have a way to translate this geometric picture to an algebraic calculation that we are able to actually do
- We have Mathematica code that computes Khovanov homology from a braid representation of a knot
- We have an algorithm to calculate the link homology in general
- We're working on a formal proof that this braiding action is a functor satisfying the braid group relations

Thank you