Quivers for 3-manifolds

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String Math, Warsaw
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Outline

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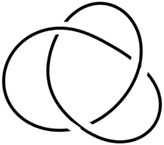
1. Knots-quivers correspondence
2. Invariants of 3-manifolds
3. Quivers for knot complements

Knots and physics

- Physical description of knots started with the famous Witten's paper about $SU(N)$ Chern-Simons theory

$$S_{CS} = \frac{k}{4\pi} \int \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$



- HOMFLY-PT polynomial of the knot K is equal to expectation value of Wilson loop operator (for SU(2) it reduces to Jones polynomial)

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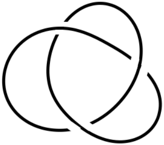

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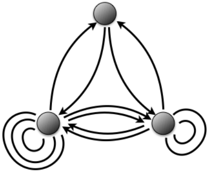
Quivers and their representations

- Quiver $Q = (Q_0, Q_1)$
  - Q_0 is a set of vertices
  - Q_1 is a set of arrows between them (loops allowed)
- Quiver representations:
  $(Q_0, Q_1) \longrightarrow$ (Vector spaces, Linear maps)
- Topological data of moduli space of representations of Q can be encoded in the motivic generating series P_Q.



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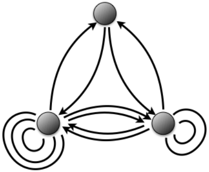

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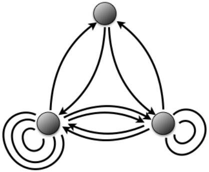
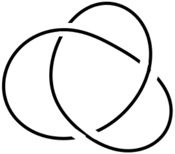
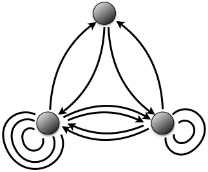
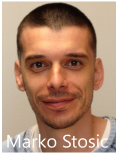


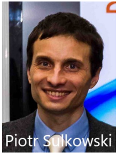
Knots-quivers correspondence

Knot K		Quiver Q
HOMFLY-PT gen. series P_K	$=$	Motivic gen. series P_Q





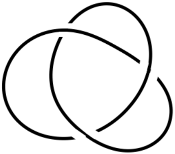


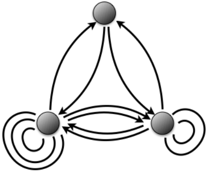


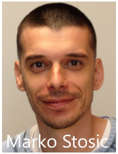
- 5-page summary: [1707.02991], full story: [1707.04017]

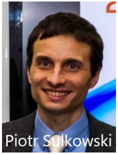
Knots-quivers correspondence

$$\frac{\text{Knot } K}{\text{HOMFLY-PT gen. series } P_K} = \frac{\text{Quiver } Q}{\text{Motivic gen. series } P_Q}$$









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Holomorphic disks/BPS particles

- Physically, knots-quivers correspondence is a duality between 3d $\mathscr{N}=2$ theories: one built from the knot and the other built from the quiver
- Geometrically, quiver nodes are holomorphic disks and arrows correspond to linking of disk boundaries


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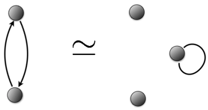

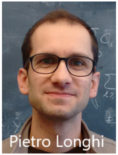

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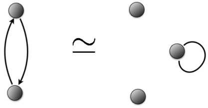

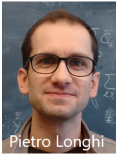

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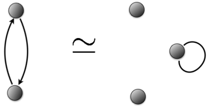

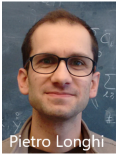

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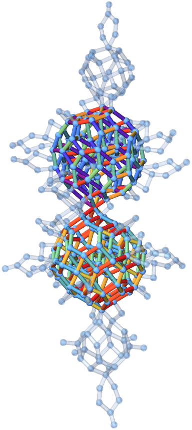
Equivalence and permutohedra

- We can study which dual quivers correspond to the same knot
- It turns out that they form intricate structures made of permutohedra!
- Details available in the paper: [2105.11806] and a poster at String Math 2022



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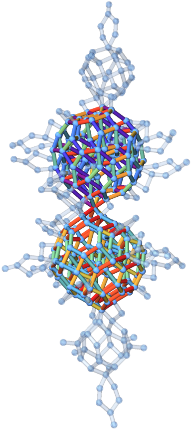

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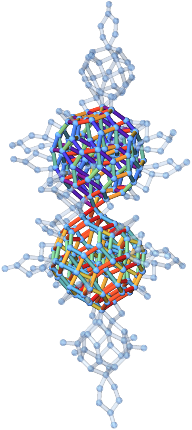

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Witten-Reshetikhin-Turaev and $\hat{Z}$ invariants

- Witten-Reshetikhin-Turaev (WRT) invariant of 3-manifold M can be expressed as a partition function of Chern-Simons theory on M
$$\mathrm{WRT}(M) = \int_M \mathscr{D}A\ e^{iS_{CS}},$$
- $\hat{Z}$ invariants are series in q with integer coefficients introduced by Gukov-Pei-Putrov-Vafa to enable categorification of WRT invariants
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$\hat{Z}$ invariants for knot complements

- We can consider a complement of the knot in the 3-sphere: $M = S^3 \backslash K$
- Gukov and Manolescu introduced a special version of $\hat{Z}$ invariants for knot complements and SU(2) gauge group:

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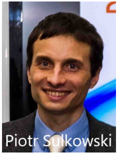
Generalisation of F_K invariants

- F_K can be generalised to $SU(N)$ [Park], so the question about $a = q^N$ (in analogy to Jones and HOMFLY-PT) arises naturally
- For $(2, 2p+1)$ torus knots explicit formulas for a-deformed F_K can be obtained directly from HOMFLY-PT polynomials
- For other knots a-deformed F_K can be constructed order by order from A-polynomials of K.
- Details available in the paper [2005.13349]



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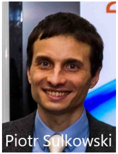

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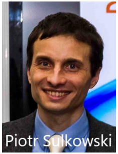

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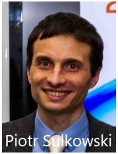

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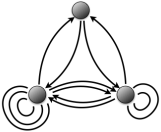
Knot complements and quivers



- I was wondering whether we can merge the knots-quivers correspondence approach with the studies on F_K invariants of knot complements.

- The answer is yes! We can define quivers for knot complements using the relation between F_K invariants and HOMFLY-PT polynomials

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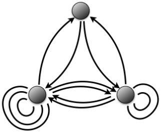

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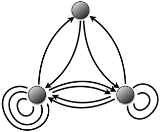

- Explicit results are given for the infinite class of $(2, 2p+1)$ torus knots complements
- We have 3d $\mathcal{N} = 2$ theories: one for the knot complement and one for the quiver
- BPS states in the knot complement theory give analogues of Labastida-Mariño-Ooguri-Vafa invariants of knots
- Details available in the paper [2005.13394]

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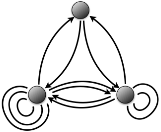

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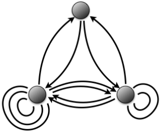

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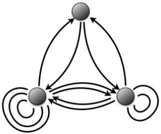

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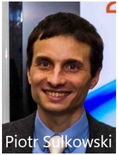
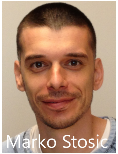
F_K invariants, quivers, and recursions

- We can directly connect quivers based on HOMFLY-PT polynomials with quivers based on F_K invariants.
- F_K invariants and their quivers correspond to branches of A-polynomials
- We can naturally define quivers for F_K invariants constructed using R-matrix approach $\rightarrow$ Sunghyuk's talk on Friday
- More details in the paper [2110.13768]





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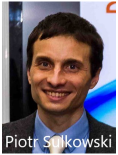

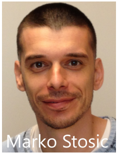

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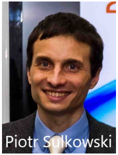

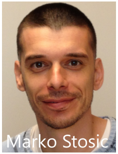

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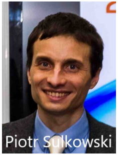

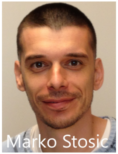

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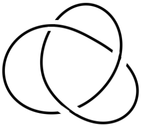 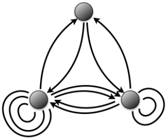

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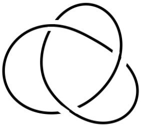
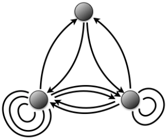
- F_K invariants can be understood as $\hat{Z}$ for the knot complement
- The correspondence with quivers can be generalised from knots to 3-manifolds using F_K invariants

Summary





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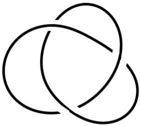

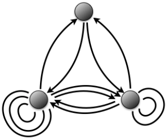

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Last message

Thank you for attention!