Path integral derivations of K-theoretic Donaldson invariants

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Based on a work to appear with J. Manschot, G. Moore, R. Tao, X. Zhang

Motivation

Five-dimensional supersymmetric gauge theories provide important playgrounds in studying interaction between geometry and physics.

- $\exists$ UV completion with 5d or 6d SCFTs [Seiberg, 96]
- They can be obtained by geometric engineering of M-theory on local Calabi-Yau threefolds. [Seiberg 96] [Morrison, Seiberg, Intriligator 96]...
- Relation to 6d QFTs or lower dimensional QFTs by KK reductions

Key tools have been various exact computations of partition functions and indices, defined on compact manifolds such as S^5, $S^4 \times S^1$.

5d SYM and K-theoretic Donaldson invariants

I will discuss correlators of 5d $\mathcal{N} = 1$ pure SU(2) gauge theory on $X \times S^1$ with a topological twist on X, when X is a smooth closed four-manifold.

$$\langle W_{\mathfrak{R}}(x_1) \cdots D(S_1) \cdots \rangle_{X \times S^1} = \int [dV] \; W_{\mathfrak{R}}(x_1) \cdots D(S_1) \cdots e^{-S[V]} \; .$$

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Mathematically, they compute the K-theoretic Donaldson invariants on X, which can be schematically written as

$$Z_\mu[\mathcal{R}, \{x_i, S_i\}] = \sum_n \mathcal{R}^{d(k)} \int_{\mathcal{M}_{k,\mu}} \hat{A}(T\mathcal{M}_{k,\mu}) \, (\mathrm{Ch}_{\mathfrak{R}}(\mathbb{E})/x_1) \cdots e^{\mu(S_1)} \cdots \; ,$$

where $\mathcal{M}_{k,\mu}$ is the moduli space of instantons on X. For complex X, this formula can be thought of as a natural 4d analogue of the Verlinde formula.

Previous works

The K-theoretic Donaldson invariants have been considered in various physical and mathematical contexts. [Nekrasov 96] [Losev, Nekrasov, Shatashvili, 98] [Göttsche, Nakajima, Yoshioka, 06] [Göttsche, Kool, Williams 19] [Hosseini, Yaakov, Zaffaroni 18] [Crichigno, Jain, Willett 18]...

However, the computations from a five-dimensional gauge theory point of view are only partially understood.

Two path integral derivations

In this talk, I will provide two different path integral derivations of the K-theoretic Donaldson invariants. This reproduces and generalizes the result of [Göttsche, Nakajima, Yoshioka, 06] and [Göttsche, Kool, Williams 19].

Two path integral derivations

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1. Coulomb branch computation ("Reduction to 4d")
   - via U-plane integral analysis
   - applicable for general class of X ($b_2^+ > 0$ and $b_1 + b_2^+$ odd)
2. Localization in $SU(2)$ gauge theory ("Reduction to 1d")
   - from the perspective of $SU(2)$ instanton counting
   - restricted to toric X
   - useful in understanding geometric interpretation of the partition function via reduction to supersymmetric quantum mechanics

5d $\mathcal{N} = 1$ gauge theories on $X \times S^1$

5d $\mathcal{N}=1$ G-gauge theory

- 5d $\mathcal{N}=1$ vector multiplet for gauge group G

$$V = (A^\mu, \sigma, \lambda^A, D^{AB})$$

- Supersymmetric YMs action

$$S_{YM} = \frac{1}{g_{\text{YM}}^2} \int d^5x \ \text{tr} \left[\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + |D_\mu \sigma|^2 + \frac{1}{2} D^{AB} D_{AB} + (\text{fermionic}) \right]$$

- Global symmetry group is $SU(2)_R \times U(1)_I$, where $U(1)_I$ is a global symmetry associated to the current

$$j = * \ \text{tr} \left(F \wedge F \right) \ ,$$

  whose charged particles are instanton particles.

- We also consider a mixed Chern-Simons term between G and $U(1)_I$

$$S_{\text{mixed CS}} = \frac{1}{8\pi^2} \int F_{(I)} \wedge \text{tr} \left(A \wedge dA + \frac{2}{3} A^3 \right) + (\text{SUSY completion})$$

BRST supercharge

We consider the theory on $X \times S^1$, with a topological twist on X,

$$[SU(2)_- \times SU(2)_+] \times SU(2)_R \quad \leftarrow \quad SU(2)_- \times SU(2)' \ .$$

This gives a BRST supercharge $\bar{\mathcal{Q}}$ such that :

$$\begin{aligned}
&\bar{\mathcal{Q}} A_\mu = \psi_\mu \ , && \bar{\mathcal{Q}} \psi_\mu = F_{\mu 5} + i D_\mu \sigma \\
&\bar{\mathcal{Q}}(iA_5 + \sigma) = 0 \ , && \mathcal{Q} \chi_{\mu\nu} = F^+_{\mu\nu} - D_{\mu\nu} \\
&\bar{\mathcal{Q}}(iA_5 - \sigma) = \eta \ , && \bar{\mathcal{Q}} \eta = D_5 \sigma \\
&\bar{\mathcal{Q}} D_{\mu\nu} = D_5 \chi_{\mu\nu} + [\sigma, \chi_{\mu\nu}] \ ,
\end{aligned}$$

with $\bar{\mathcal{Q}}^2 \sim \partial_5$. This procedure gives a (partial) topological theory on $X \times S^1$.

The topological reduction on X gives a 1d $\mathcal{N} = 1$ supersymmetric quantum mechanics on S^1.

What does the partition function compute?

Reduction to SQM and moduli space of instantons

The effective quantum mechanics on S^1 is the 1d $\mathcal{N} = 1$ "sigma-model" into the moduli space of instantons on X,

$$S^1 \quad \to \quad \bigsqcup_{k=0}^{\infty} \mathcal{M}_{\mu,k} \ ,$$

where $\mathcal{M}_{\mu,k}$ is the moduli space with instanton number k and $w_2(P) = \mu$. The Hilbert space of the QM is the space of sections of the spin bundle $\mathcal{S}$ on $\mathcal{M}_{\mu,k}$,

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_{\mu,k} \ .$$

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The twisted partition function can be written as an intersection integral over $\mathcal{M}_{\mu,k}$ [Nekrasov 96]

$$\begin{aligned} Z_\mu[\mathcal{R}] &= \sum_{k=0}^{\infty} \mathcal{R}^{d(k)} \ \mathrm{Tr}_{\mathcal{H}_{\mu,k}}(-1)^F \\ &= \sum_{k=0}^{\infty} \mathcal{R}^{d(k)} \int_{\mathcal{M}_{\mu,k}} \hat{A}(T\mathcal{M}_{\mu,k}) \ , \end{aligned}$$

where $d(k) = 2kh_G^\vee - \dim(G)(\chi+\sigma)/4$.

Topological observables

- Wilson loop along S^1

$$W_{\mathfrak{R}}(x) = \mathrm{Tr}_{\mathfrak{R}}\ \mathrm{P}\exp\int_{S^1}(iA_5+\sigma)dx^5\ ,$$

  where $x \in H_0(X)$.

- Topological descents of Wilson loop

- 3D Chern-Simons observable on $S^1 \times S$, where $S \in H_2(X)$ [Baulieu, Losev, Nekrasov 97]

$$S_{\text{mixed CS}} = \int_{X\times S^1} F_{(I)} \wedge \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A^3\right) + (\text{SUSY completion})\ ,$$

  for a closed two-form $F_{(I)} = PD(S^1 \times S)$.

- Other 3D TFTs

3D CS observable and and Donaldson map

Consider the 3D Chern-Simons observable,

$$S_{\text{mixed CS}}[V] = \frac{1}{4\pi^2}\int F_{(I)} \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A^3\right) + (\text{SUSY completion}) \ ,$$

with $\left[F_{(I)}/2\pi\right] = \mathbf{n} \in H^2(X, \mathbb{Z})$.

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This induces a line bundle $\mathcal{L}_I$ on the moduli space with

$$c_1(\mathcal{L}_I) = \frac{1}{4\pi^2}\int_{M_4} F_{(I)} \wedge \mathrm{Tr}\left(\mathbb{F} \wedge \mathbb{F}\right) \ ,$$

where $\mathbb{F}$ is the curvature of the universal bundle over $X \times \mathcal{M}_{\mu,k}$. The Hilbert space of QM becomes the space of section of $\mathcal{S} \otimes \mathcal{L}_I$, and

$$Z_\mu[\mathcal{R}, \mathbf{n}] = \sum_{k=0}^{\infty} \mathcal{R}^{d(k)} \int_{\mathcal{M}_{\mu,k}} \hat{A}(T\mathcal{M}_{\mu,k}) \wedge \exp c_1(\mathcal{L}) \ .$$

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Note that this factor can be interpreted as the insertion of $\exp \mu(S)$, which is the Donaldson map

$$\mu : H_2(X) \to H^2(\mathcal{M}) \ ,$$

evaluated at the co-dimension two locus $S = PD(\mathbf{n}) \in H_2(X)$.

Wilson loop

We can also consider a Wilson loop at $x \in H_0(X)$.

$$W_{\mathfrak{R}}(x) = \mathrm{Tr}_{\mathfrak{R}}\ \mathrm{P}\exp \int_{S^1} (iA_5 + \sigma) dx^5 \ .$$

This leads to the insertion of the Chern character of the universal bundle $\mathbb{E}$ in representation $\mathfrak{R}$,

$$\mathrm{Ch}_{\mathfrak{R}}(\mathbb{E})/x \ ,$$

where $/ : H^\bullet(\mathcal{M}_{\mu,k} \times X) \times H_i(X) \to H^{\bullet - i}(\mathcal{M})$.

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To summarize,

$$Z_\mu[\mathcal{R}, \{x_{\mathfrak{R}}, S\}] = \sum_{k=0}^{\infty} \mathcal{R}^{d(k)} \int_{\mathcal{M}_{\mu,k}} \hat{A}(T\mathcal{M}_{\mu,k})\,(\mathrm{Ch}_{\mathfrak{R}}(\mathbb{E})/x)\exp\mu(S)\ .$$

Strategy

Our approach to computation of the partition functions on $X \times S^1$:

- The 5d $\mathcal{N} = 1$ theory on S^1 can be thought of as an effective 4d $\mathcal{N} = 2$ theory on X, with infinitely many Kaluza-Klein particles with mass $m_{\mathrm{KK}} \sim n/R$, for $n \in \mathbb{Z}$.

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- Using the topological invariance on X, we scale the metric on X as

$$g_{\mu\nu} \to t g_{\mu\nu} \ , \qquad t \to \infty.$$

- The physics is replaced by a low energy effective theory on the Coulomb branch of 4d $\mathcal{N} = 2$ theory.
- The Coulomb branch is described by the Seiberg-Witten geometry.
- Computation of the partition function is reformulated into that of Seiberg-Witten invariants, with contributions from the U-plane integral.

Seiberg-Witten Geometry of Effective 4d $\mathcal{N} = 2$ Theory

Coulomb branch of 4d effective theory

- Let us focus on $G = SU(2)$. Classical Coulomb branch of effective 4d theory is parametrized by

$$a = \frac{1}{R} \int_{S^1} (\sigma + iA_5)\, dx^5 \qquad \in \mathbb{R} \times S^1 \ .$$

- The Coulomb branch effective theory is determined by the prepotential

$$F(a, \Lambda) = \frac{2}{R^2} \left(Li_3(e^{-Ra}) - \zeta(3) \right) + a^2 \left(\log(\mathcal{R}) - \frac{\pi i}{2} \right) + \mathcal{O}(R\Lambda) \ ,$$

where Λ is a dynamically generated scale of the 4d theory, which is related to the 5d gauge coupling

$$4 \log(R\Lambda) = -\frac{8\pi^2 R}{g_{\mathrm{YM}}^2}$$

It is useful to define a dimensionless parameter $\mathcal{R} = R\Lambda$, which is the instanton counting parameter.

Seiberg-Witten curve

The order parameter for the quantum corrected Coulomb branch is the VEV of the fundamental Wilson loop,

$$U = \langle \mathrm{Tr}_F \ \mathrm{P} \exp(Ra) \rangle = e^{Ra} + e^{-Ra} + \mathcal{O}(R\Lambda) \ .$$

The Seiberg-Witten curve is parametrized by $U \in \mathbb{C}$, [Nekrasov 96] [Ganor, Morrison, Seiberg 96] [Eguchi, Sakai 02]

$$-\mathcal{R}^2 X \left(\omega + \frac{1}{\omega} \right) = P(X)^2 \ , \qquad P(X) = X^2 + UX + 1 \ .$$

with

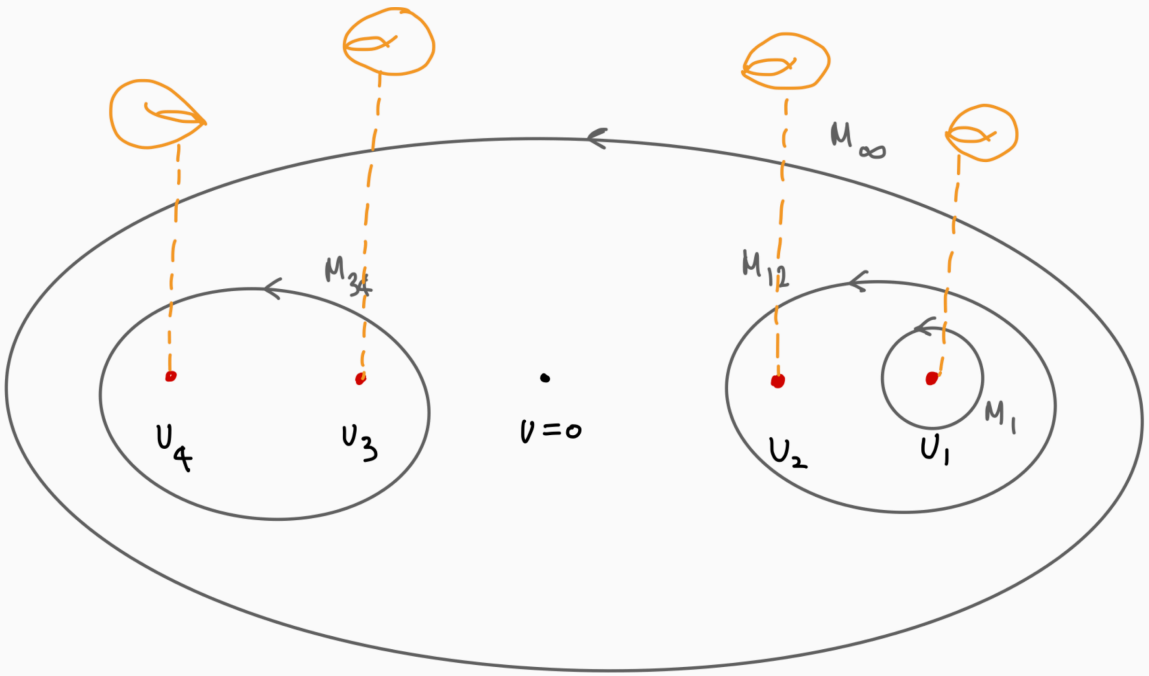
$$\lambda = \frac{1}{2\pi i R} \log(X) \frac{d\omega}{\omega} \ .$$

Then

$$a = \int_A \lambda \ , \qquad a_D = \int_B \lambda \ .$$

U-plane

Seiberg-Witten geometry is an elliptic fibration over the U-plane. At generic value of $\mathcal{R}$, it is parametrized by $U \in \mathbb{C}$ with four singularities, where a BPS particle becomes massless.



U_1 : A monopole becomes massless $\qquad$ U_2: A dyon becomes massless

U_3, U_4 : A dyonic instanton becomes massless

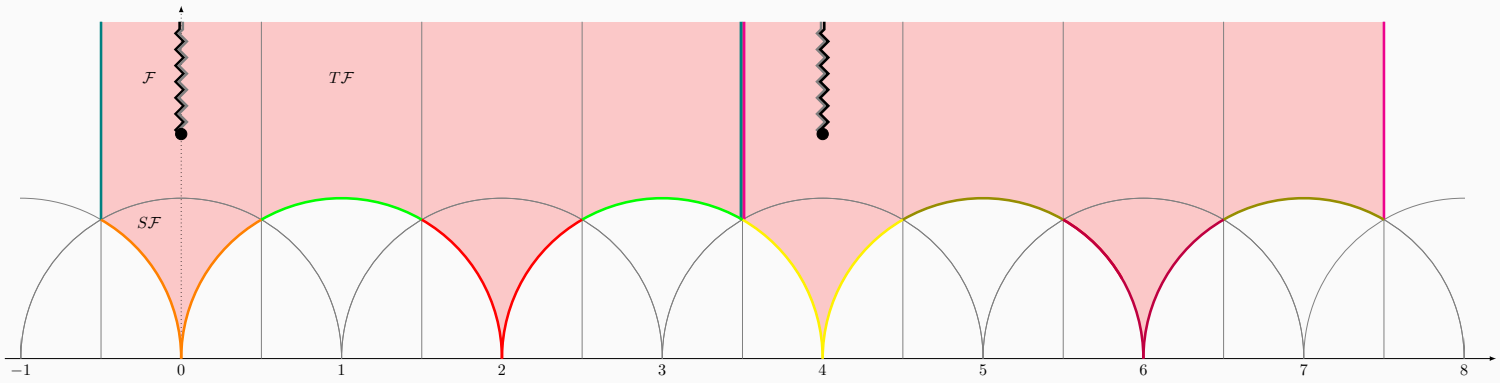
See also [Closset-Magureanu, 21]

Fundamental domain of τ

From the Seiberg-Witten curve, we obtain the relation

$$U(\tau) = \pm \left(-4\mathcal{R}^2 \frac{\theta_2(\tau)^4 + \theta_3(\tau)^4}{\theta_2(\tau)^2 \theta_3(\tau)^2} + 4\mathcal{R}^4 + 4 \right)^{1/2} .$$

- For $\mathcal{R} = 1$, $U(\tau)$ is modular invariant for $\Gamma^0(8) \subset SL(2,\mathbb{Z})$. Therefore, the fundamental domain is $\mathcal{F}_{\mathcal{R}=1} = \mathbb{H}/\Gamma^0(8)$.
- For generic $\mathcal{R}$, $\exists$ branch points at $U(\tau) = 0$, whose position depends on $\mathcal{R}$.



Fundamental domain $\mathcal{F}_R$, which is a double copy of $\mathbb{H}/\Gamma^0(4)$ of 4d SYM.

See also [Aspman, Furrer, Manschot, 21]

U-plane integral

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Following [Moore, Witten 97], the partition function of effective 4d theory for $b_2^+(X) > 0$ can be written as

$$Z_{J,\mu}[\mathcal{R}, \mathbf{n}] = \Phi_{J,\mu}[\mathcal{R}, \mathbf{n}] + \sum_{i=1}^{4} Z^{SW}_{J,\mu,i}[\mathcal{R}, \mathbf{n}] \ ,$$

where Φ is the so-called "U-plane integral" contribution and Z^{SW} is the Seiberg-Witten contribution at the four singular points in the U-plane, where a BPS particle becomes massless.

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- For $b_2^+ > 1$, $Z_{J,\mu}$ is independent of metric on X.
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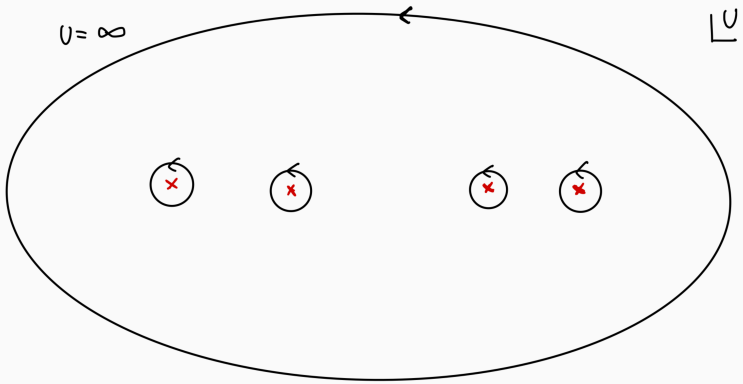
- For $b_2^+ > 1$, $Z_{J,\mu}$ is independent of metric on X.
- For $b_2^+ > 1$, $\Phi_{J,\mu}$ identically vanishes.
- For $b_2^+ = 1$, the partition functions are expected to jump discontinuously as a function of metric on X.
- The metric dependence comes through J, the period point. $J \in H^2(X, \mathbb{R})$ with $J = *J$ and $J^2 = 1$).

U-plane integral for $b_2^+ = 1$

$$Z_{J,\mu} = \Phi_{J,\mu} + \sum_{i=1}^{4} Z^{SW}_{J,\mu,i}$$



- For $b_2^+ = 1$, $Z_{J,\mu}$ is a piecewise constant function of J. The dependence on J only comes from the region $U \to \infty$.
- The J-dependence of Φ around the singularities at finite $U = U_i$ are also non-trivial, but they are canceled with the wall-crossing of $Z^{SW}_{J,\mu,i}$.
- We can utilize this fact to compute $Z^{SW}_{J,\mu,i}$ for $b_2^+ > 1$.

Let us focus on $b_2^+ = 1$ first.

Coulomb branch effective action

- The effective theory on Coulomb branch can be thought of as a $\mathcal{N}=2$ $U(1)_G \times U(1)_I$ theory.
- We assume $b_1 = 0$ and $b_2^+ = 1$. Then only the zero mode contributes to the U-plane integral. [Moore-Witten 97] The Coulomb branch effective action restricted to zero modes reads

$$\begin{aligned} S &= \int_X \frac{i}{16\pi}\left(\bar{\tau}_{ab}F_+^a \wedge F_+^b + \tau_{ab}F_-^a \wedge F_-^b\right) - \frac{1}{8\pi}\mathrm{Im}(\tau_{ab})D^a \wedge D^b \\ &\quad + \frac{i\sqrt{2}}{16\pi}\bar{F}_{abc}\eta^a\chi^b \wedge (D+F_+)^c \ . \end{aligned}$$

- The couplings τ_{ab} for $a = 1, 2$ are

$$\tau = -\frac{1}{2\pi i}\frac{\partial^2 F}{\partial a^2} \ , \quad v = -\frac{1}{2\pi i}\frac{\partial^2 F}{\partial a \partial m_I} \ , \quad \xi = -\frac{1}{2\pi i}\frac{\partial^2 F}{\partial m_I^2} \ ,$$

where $m_I = \frac{2}{R}\log(\mathcal{R})$.

Integral over $\mathcal{F}_R$

Integrating the zero modes, the U-plane integral becomes an integral over $\mathcal{F}_{\mathcal{R}}$:

$$\Phi_{J,\mu}[\mathcal{R},\mathbf{n}] = K_U \int_{\mathcal{F}_R} d\tau \wedge d\bar{\tau}\ \nu_R(\tau) C^{\mathbf{n}^2} \Psi^J_\mu(\tau,\bar{\tau}, v\mathbf{n}/2, \bar{v}\mathbf{n}/2).$$

From the Seiberg-Witten geometry, the couplings can be written as q-series, where $q = e^{2\pi i\tau}$ [Göttsche, Nakajima, Yoshioka]

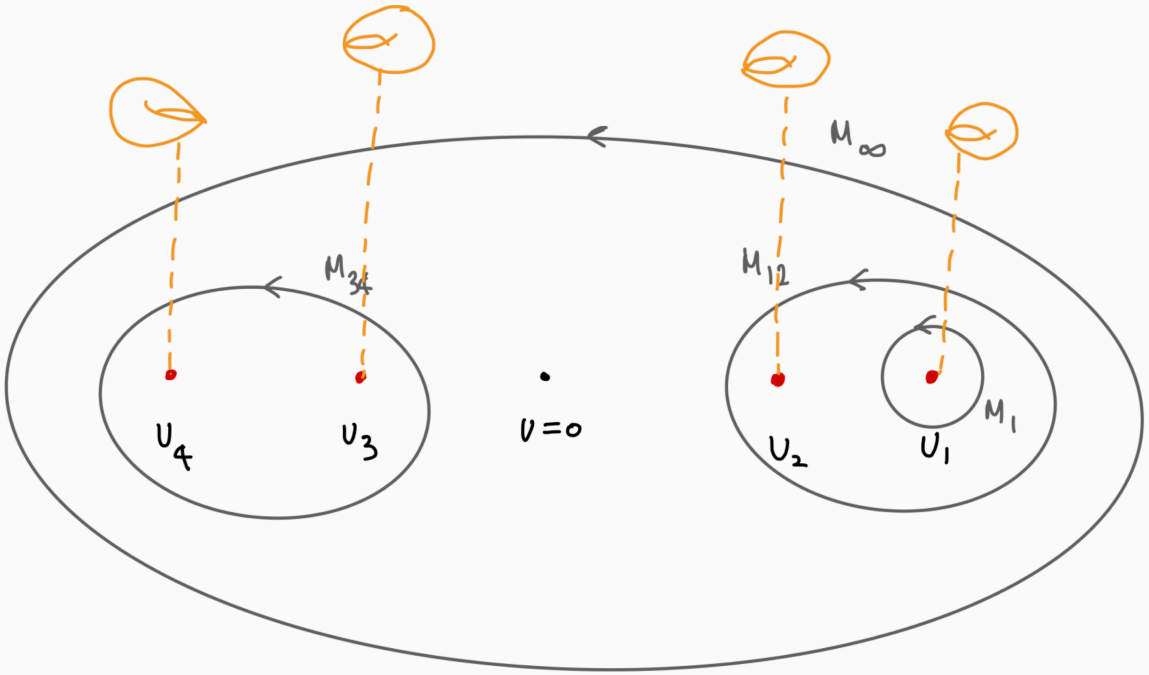
$$\frac{\theta_1(\tau, v/2)}{\theta_4(\tau, v/2)} = -\mathcal{R}\ , \qquad C = e^{2\pi i\xi} = \frac{\theta_4(\tau, v/2)}{\theta_4(\tau)}\ ,$$

and

$$\begin{aligned}\Psi^J_\mu(\tau,\bar{\tau},z,\bar{z}) \quad &= \quad e^{-2\pi y b_+^2} \sum_{k\in H^2(X,\mathbb{Z})+\mu} \partial_{\bar{\tau}}(\sqrt{2y}B(k+b,J)) \\ &\quad \cdot(-1)^{B(k,\mathcal{K}_X)} q^{-k_-^2/2} \bar{q}^{k_+^2/2} e^{-2\pi i B(k_-,z)-2\pi i B(k_+,\bar{z})}\ ,\end{aligned}$$

with $b = \mathrm{Im}\ \mathrm{z}/y$. We defined $B(k_1,k_2) = \int_X k_1 \wedge k_2$.

Monodromies



- It is highly non-trivial that the integrand is single-valued under various monodromies in the U-plane.
- A proper choice of $\bar{\mathcal{Q}}$-exact terms is crucial in establishing the single-valuedness.
- This condition determines further quantization of the background flux $\mathbf{n}$.

Wall-crossing

Suppose that we have two period points J and J'. Following [Korpas, Manschot 17] [Korpas, Manschot, Moore, Nidaiev 19], one can show

$$\Psi_\mu^J(\tau,\bar\tau,z,\bar z) - \Psi_\mu^{J'}(\tau,\bar\tau,z,\bar z) = \partial_{\bar\tau}\widehat{\Theta}_\mu^{J,J'}(\tau,\bar\tau,z,\bar z)\ ,$$

$$\begin{aligned}\widehat{\Theta}_\mu^{J,J'}(\tau,\bar\tau,z,\bar z) \quad = \quad & \sum_{k\in H^2(X,\mathbb{Z})+\mu} \frac{1}{2}\left[E(\sqrt{2y}B(k+b,J)) - E(\sqrt{2y}B(k+b,J'))\right] \\ & (-1)^{B(k,\mathcal{K}_X)} q^{-k^2/2} e^{-2\pi i B(k,z)}\ ,\end{aligned}$$

where $E(x) = \mathrm{Erf}(\sqrt{\pi}x)$ and $y = \mathrm{Im}(\tau)$, $b = \mathrm{Im}(z)/y$.

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where $E(x) = \mathrm{Erf}(\sqrt{\pi}x)$ and $y = \mathrm{Im}(\tau)$, $b = \mathrm{Im}(z)/y$.

From this, we can write down the wall-crossing formula

$$Z_\mu^J[\mathcal{R},\mathbf{n}] - Z_\mu^{J'}[\mathcal{R},\mathbf{n}] = \sum_{k\in\mathcal{W}_{J,J'}} 8\left[\nu_R(\tau) C^{\mathbf{n}^2}(-1)^{B(k,K)} q^{-k^2/2} e^{-2\pi i B(k,\mathbf{n}v/2)}\right]_{q^0}\ ,$$

$$\text{where}\quad \mathcal{W}_{J,J'} = \left\{k \mid B(k-\mathbf{n}/4,J) > 0 \ \text{ and } B(k-\mathbf{n}/4,J') < 0\ \right\}\ .$$

Comparison to GNY

There are two ways to evaluate this expression.

(1) Expand the integrand in small $\mathcal{R}$ first and evaluate the q^0 term. This procedure reproduces the formula of [Göttsche, Nakajima, Yoshioka 06]. For example, $Z_\mu[\mathbf{n}, \mathcal{R}]$ for $X = \mathbb{P}^2$ can be computed via the WC formula, using a blowup and existence of vanishing chamber.

$$Z_0^{\mathbb{P}^2}[\mathbf{n}, \mathcal{R}] = \begin{cases} \mathcal{R} + \mathcal{R}^5 + \mathcal{R}^9 + \cdots & \mathbf{n} = 0 \\ 3\mathcal{R} + 6\mathcal{R}^5 + 10\mathcal{R}^9 + \cdots & \mathbf{n} = 1 \\ 6\mathcal{R} + 21\mathcal{R}^5 + 56\mathcal{R}^9 & \mathbf{n} = 2 \\ 10\mathcal{R} + 56\mathcal{R}^5 + 230\mathcal{R}^9 + \cdots & \mathbf{n} = 3 \\ \cdots \end{cases}$$

which agrees with the interpretation

$$Z_0^{\mathbb{P}^2}[\mathbf{n}, \mathcal{R}] = \sum_{d \geq 0} \chi\left(\mathcal{M}_{0,d}^{\mathbb{P}^2}, \mathcal{O}(\mu(H^{\otimes \mathbf{n}}))\right) \mathcal{R}^d \ .$$

A puzzle

However, in the five-dimensional theory point of view, it is more natural to

(2) Keep $\mathcal{R}$ finite and expand the integrand in small q first, and take the q^0 term. Surprisingly, we find that the result does not agree with (1). In particular, for $b_2^+ = 1$, the result of the U-plane integral contains negative powers of $\mathcal{R}$.

A puzzle : How do we interpret the $\mathcal{R}^{-1}$ dependence of the partition function?

Toric localization

Toric localization

There is an alternative path integral derivation when X is a smooth toric four-manifold.

Using the toric action, Z can be written as a sum over contributions from the instantons localized at fixed loci on X. Schematically,

$$Z \sim \sum_k \int da \prod_{i=1}^{\chi} Z(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda^i) \ ,$$

where $Z(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda^i)$ is the K-theoretic Nekrasov partition function on $S^1 \times \mathbb{C}^2_{\epsilon_1^i \epsilon_2^i}$, localized at i-th fixed locus on X.

This is the approach adopted by many authors including [Nekrasov] [GNY] [Hosseini et al.] [Crichigno et al.] [Bonelli et al.]...

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Remaining questions are

- What is the contour?
- Comparison with the U-plane integral approach?

Non-equivariant limit

Let us focus on the non-equivariant limit $\epsilon_1, \epsilon_2 \to 0$ of the partition function. The Nekrasov conjecture tells us that

$$\begin{aligned}\log Z(a, \epsilon_1, \epsilon_2, R, \Lambda) &= \frac{1}{\epsilon_1 \epsilon_2} F(a, R, \Lambda) + \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} H(a, \Lambda) \\ &\quad + \log A(a, \Lambda, R) + \frac{\epsilon_1^2 + \epsilon_2^2}{3\epsilon_1 \epsilon_2} \log B(a, \Lambda, R) + \cdots\end{aligned}$$

Summing over contributions from all the fixed loci, we obtain the finite non-equivariant limit, the integrand becomes

$$\begin{aligned}Z_{k,\mu}(a, \mathcal{R}, \mathbf{n}) &= \prod_{i=1}^{\chi} Z(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda^i) \\ &= \exp\left[\frac{1}{2}\frac{\partial^2 F}{\partial a^2} k^2 + \frac{R}{4}\frac{\partial^2 F}{\partial a \partial \log \mathcal{R}} k\mathbf{n} + \frac{R^2}{32}\frac{\partial^2 F}{\partial \log \mathcal{R}^2}\mathbf{n}^2 \right. \\ &\quad \left. + \frac{\partial H}{\partial a} k\mathcal{K}_X + \chi A + \sigma B\right] + \mathcal{O}(\epsilon_1, \epsilon_2)\ .\end{aligned}$$

Contour encodes the metric dependence

Note that $b_2^+ = 1$ for a toric four-manifold. The partition function undergoes a wall-crossing.

The supersymmetric localization process gives

$$\begin{aligned} Z_{J,\mu}[\mathcal{R}, \mathbf{n}] &= \sum_k \int \frac{dh}{h} \int_{\mathfrak{M}} da d\bar{a}\ \partial_{\bar{a}} Z_{k,\mu,J}(a, \bar{a}, 0, 0, h) \\ &= \sum_k \oint_{C_J} da\ Z_{k,\mu}(a)\ . \end{aligned}$$

The contour C_J depends on the choice of metric, J. After a careful analysis of the zero mode integrals, we arrive at the wall-crossing formula in this approach,

$$Z_{J,\mu}[\mathcal{R}, \mathbf{n}] - Z_{J',\mu}[\mathcal{R}, \mathbf{n}] = \sum_{k \in \mathcal{W}_{J,J'}} \left(\operatorname*{res}_{a=\infty} da + \operatorname*{res}_{a=-\infty} da \right) Z_{k,\mu}(a, \mathcal{R}, \mathbf{n})\ ,$$

$$\text{where} \quad \mathcal{W}_{J,J'} = \left\{ k \mid B(k - \mathbf{n}/4, J) > 0 \ \text{ and } B(k - \mathbf{n}/4, J') < 0 \right\}\ .$$

This agrees precisely with the WC formula in the U-plane integral approach.

$b_2^+(X) > 1$ and the Seiberg-Witten contribution

K-theoretic Donaldson invariants for $b_2^+(X) > 1$

From

$$Z_{J,\mu} = \Phi_{J,\mu} + \sum_{i=1}^{4} Z^{SW}_{J,\mu,i} \ ,$$

and the fact that the wall-crossing at the strong coupling singularities cancels between Φ and Z^{SW},

$$\Phi_{J,\mu}|_{U_i} - \Phi_{J',\mu}|_{U_i} = -\left(Z^{SW}_{J,\mu,i} - Z^{SW}_{J',\mu,i}\right) \ ,$$

it is possible to derive the K-theoretic Donaldson invariants for $b_2^+(X) > 1$. We find

$$Z_\mu[\mathcal{R}, \mathbf{n}] = \sum_{l \bmod 4} e^{-i\pi(3\chi_h/2+2\mu^2)l} G_\mu(e^{-i\pi l/2}\mathcal{R}, \mathbf{n}) \ ,$$

where

$$G_\mu(\mathcal{R}, \mathbf{n}) = \frac{2^{2\chi+3\sigma-\chi_h}}{(1-\mathcal{R}^2)^{\mathbf{n}^2/2+\chi_h}} \sum_c \mathrm{SW}(c) \left(\frac{1-\mathcal{R}}{1+\mathcal{R}}\right)^{B(\mathbf{n},c)/2} e^{\pi i B(K+c,\mu)} \ .$$

This reproduces the result of [Göttsche, Kool, Williams 19].

Conclusion

Summary

- The topological correlator of 5d $\mathcal{N} = 1$ $SU(2)$ gauge theory computes the K-theoretic Donaldson invariant

$$Z_{J,\mu}[\mathcal{R}, \{x, S\}] = \sum_k \mathcal{R}^{d(k)} \int_{\mathcal{M}_{k,\mu}} \hat{A}_R(T\mathcal{M}_{k,\mu}) \left(\mathrm{Ch}_R(\mathbb{E})/x\right) e^{\mu(S)} \ .$$

- In particular, we derived the metric dependence of $Z_{J,\mu}$ for $b_1 = 0$ and $b_2^+ = 1$ in the perspective of the U-plane integral, and also in the $SU(2)$ gauge theory perspective. We show that they agree in a non-trivial way.
- Using the U-plane integral approach, the computation can be generalized to $b_2^+ > 1$. This reproduces the result of [Göttsche, Kool, Williams 19]
- The appearance of negative powers of $\mathcal{R}$ for $b_2^+(X) = 1$ remains to be understood.

Future directions

- Extension to general rank 1 E_N theories and 5d $\mathcal{N} = 1^*$ theory. The latter is expected to compute the χ_y-genus of the moduli space, which is identified with the K-theoretic Vafa-Witten invariants.
- Using the U(1) Kaluza-Klein symmetry, it is possible to compute the partition functions on more general five-manifolds (e.g., S^5) via the U-plane integral approach.
- Uplift to 6d $\mathcal{N} = (1, 0)$ gauge theories. It is expected to compute the "elliptic invariants" on the moduli space of instantons.
- Extension to $b_1(X) > 0$ and dimensional reduction to 3d $\mathcal{N} = 4$ theories.
- More general topological defects.