Path integral derivations of K-theoretic Donaldson invariants

Heeyeon Kim (Rutgers)

Based on a work to appear with J. Manschot, G. Moore, R. Tao, X. Zhang
Five-dimensional supersymmetric gauge theories provide important playgrounds in studying interaction between geometry and physics.

- ∃ UV completion with 5d or 6d SCFTs [Seiberg, 96]
- They can be obtained by geometric engineering of M-theory on local Calabi-Yau threefolds. [Seiberg 96] [Morrison, Seiberg, Intriligator 96]...
- Relation to 6d QFTs or lower dimensional QFTs by KK reductions

Key tools have been various exact computations of partition functions and indices, defined on compact manifolds such as $S^5$, $S^4 \times S^1$. 
I will discuss correlators of 5d $\mathcal{N} = 1$ pure SU(2) gauge theory on $X \times S^1$ with a topological twist on $X$, when $X$ is a smooth closed four-manifold.

$$\langle W_{\mathfrak{r}}(x_1) \cdots D(S_1) \cdots \rangle_{X \times S^1} = \int [dV] \ W_{\mathfrak{r}}(x_1) \cdots D(S_1) \cdots e^{-S[V]}.$$
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\]

Mathematically, they compute the **K-theoretic Donaldson invariants** on $X$, which can be schematically written as

\[
Z_\mu[\mathcal{R}, \{x_i, S_i\}] = \sum_n \mathcal{R}^{d(k)} \int_{\mathcal{M}_{k,\mu}} \hat{A}(T\mathcal{M}_{k,\mu}) (\text{Ch}_{\mathcal{R}}(E)/x_1) \cdots e^{\mu(S_1)} \cdots ,
\]

where $\mathcal{M}_{k,\mu}$ is the **moduli space of instantons** on $X$. For complex $X$, this formula can be thought of as a natural 4d analogue of the Verlinde formula.
Previous works

The K-theoretic Donaldson invariants have been considered in various physical and mathematical contexts. [Nekrasov 96] [Losev, Nekrasov, Shatashvili, 98] [Göttsche, Nakajima, Yoshioka, 06] [Göttsche, Kool, Williams 19] [Hosseini, Yaakov, Zaffaroni 18] [Crichigno, Jain, Willett 18]...

However, the computations from a five-dimensional gauge theory point of view are only partially understood.
In this talk, I will provide two different path integral derivations of the K-theoretic Donaldson invariants. This reproduces and generalizes the result of [Göttsche, Nakajima, Yoshioka, 06] and [Göttsche, Kool, Williams 19].
In this talk, I will provide two different path integral derivations of the K-theoretic Donaldson invariants. This reproduces and generalizes the result of [Göttsche, Nakajima, Yoshioka, 06] and [Göttsche, Kool, Williams 19].

1. **Coulomb branch computation** ("Reduction to 4d")
   - via U-plane integral analysis
   - applicable for general class of $X$ ($b_2^+ > 0$ and $b_1 + b_2^+$ odd)

2. **Localization in $SU(2)$ gauge theory** ("Reduction to 1d")
   - from the perspective of $SU(2)$ instanton counting
   - restricted to toric $X$
   - useful in understanding geometric interpretation of the partition function via reduction to supersymmetric quantum mechanics
5d $\mathcal{N} = 1$ gauge theories on $X \times S^1$
5d $\mathcal{N} = 1$ G-gauge theory

- 5d $\mathcal{N} = 1$ vector multiplet for gauge group $G$
  
  $V = (A^\mu, \sigma, \lambda^A, D^{AB})$

- Supersymmetric YMs action
  
  $$S_{YM} = \frac{1}{g_{YM}^2} \int d^5x \, \text{tr} \left[ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + |D_\mu \sigma|^2 + \frac{1}{2} D^{AB} D_{AB} + \text{(fermionic)} \right]$$

- Global symmetry group is $SU(2)_R \times U(1)_I$, where $U(1)_I$ is a global symmetry associated to the current
  
  $$j = * \text{tr} (F \wedge F),$$

  whose charged particles are instanton particles.

- We also consider a mixed Chern-Simons term between $G$ and $U(1)_I$
  
  $$S_{\text{mixed CS}} = \frac{1}{8\pi^2} \int F_{(I)} \wedge \text{tr} \left( A \wedge dA + \frac{2}{3} A^3 \right) + \text{(SUSY completion)}$$
We consider the theory on $X \times S^1$, with a topological twist on $X$,

$$\left[ SU(2)_- \times SU(2)_+ \right] \times SU(2)_R \leftarrow SU(2)_- \times SU(2)' .$$

This gives a BRST supercharge $\bar{Q}$ such that:

$$\bar{Q} A_\mu = \psi_\mu , \quad \bar{Q} \psi_\mu = F_{\mu 5} + i D_\mu \sigma$$

$$\bar{Q} (i A_5 + \sigma) = 0 , \quad Q \chi_{\mu \nu} = F^+_{\mu \nu} - D_{\mu \nu}$$

$$\bar{Q} (i A_5 - \sigma) = \eta , \quad \bar{Q} \eta = D_5 \sigma$$

$$\bar{Q} D_{\mu \nu} = D_5 \chi_{\mu \nu} + [\sigma, \chi_{\mu \nu}] ,$$

with $\bar{Q}^2 \sim \partial_5$. This procedure gives a (partial) topological theory on $X \times S^1$.

The topological reduction on $X$ gives a $1d \; \mathcal{N} = 1$ supersymmetric quantum mechanics on $S^1$. 
What does the partition function compute?
Reduction to SQM and moduli space of instantons

The effective quantum mechanics on $S^1$ is the 1d $\mathcal{N} = 1$ “sigma-model” into the **moduli space of instantons** on $X$,

\[ S^1 \rightarrow \bigsqcup_{k=0}^{\infty} \mathcal{M}_{\mu,k}, \]

where $\mathcal{M}_{\mu,k}$ is the moduli space with instanton number $k$ and $w_2(P) = \mu$. The Hilbert space of the QM is the space of sections of the spin bundle $S$ on $\mathcal{M}_{\mu,k}$,

\[ \mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_{\mu,k}. \]
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The twisted partition function can be written as an intersection integral over $\mathcal{M}_{\mu,k}$ [Nekrasov 96]

$$Z_\mu[\mathcal{R}] = \sum_{k=0}^{\infty} \mathcal{R}^{d(k)} \operatorname{Tr}_{\mathcal{H}_{\mu,k}} (-1)^F$$

$$= \sum_{k=0}^{\infty} \mathcal{R}^{d(k)} \int_{\mathcal{M}_{\mu,k}} \mathcal{A}(T\mathcal{M}_{\mu,k}),$$

where $d(k) = 2kh_G^\vee - \dim(G)(\chi + \sigma)/4$. 

Topological observables

- Wilson loop along $S^1$
  \[ W_{\mathfrak{R}}(x) = \text{Tr}_{\mathfrak{R}} \ P \exp \int_{S^1} (iA_5 + \sigma) dx^5, \]
  where $x \in H_0(X)$.

- Topological descents of Wilson loop

- 3D Chern-Simons observable on $S^1 \times S$, where $S \in H_2(X)$ [Baulieu, Losev, Nekrasov 97]
  
  \[ S_{\text{mixed CS}} = \int_{X \times S^1} F_{(I)} \wedge \text{Tr} \left( A \wedge dA + \frac{2}{3} A^3 \right) + (\text{SUSY completion}) , \]
  for a closed two-form $F_{(I)} = PD(S^1 \times S)$.

- Other 3D TFTs
Consider the 3D Chern-Simons observable,

\[ S_{\text{mixed CS}}[V] = \frac{1}{4\pi^2} \int F_{(I)} \wedge \text{Tr} \left( A \wedge dA + \frac{2}{3} A^3 \right) + \text{(SUSY completion)}, \]

with \( [F_{(I)}/2\pi] = n \in H^2(X, \mathbb{Z}) \).
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with \([F(I)/2\pi] = n \in H^2(X, \mathbb{Z}).\)

This induces a line bundle \(L_I\) on the moduli space with

\[ c_1(L_I) = \frac{1}{4\pi^2} \int_{M_4} F(I) \wedge \text{Tr} (\mathbb{F} \wedge \mathbb{F}) , \]

where \(\mathbb{F}\) is the curvature of the universal bundle over \(X \times \mathcal{M}_{\mu,k}\). The Hilbert space of QM becomes the space of section of \(S \otimes L_I\), and

\[ Z_{\mu}[\mathcal{R}, n] = \sum_{k=0}^{\infty} \mathcal{R}^{d(k)} \int_{\mathcal{M}_{\mu,k}} \hat{A}(T\mathcal{M}_{\mu,k}) \wedge \exp c_1(L) . \]
Consider the 3D Chern-Simons observable,

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This induces a line bundle \( \mathcal{L}_I \) on the moduli space with

\[ c_1(\mathcal{L}_I) = \frac{1}{4\pi^2} \int_{M_4} F(I) \wedge \text{Tr} (\bar{F} \wedge F) , \]

where \( \bar{F} \) is the curvature of the universal bundle over \( X \times \mathcal{M}_{\mu,k} \). The Hilbert space of QM becomes the space of section of \( S \otimes \mathcal{L}_I \), and

\[ Z_\mu[R, n] = \sum_{k=0}^{\infty} R^{d(k)} \int_{\mathcal{M}_{\mu,k}} \hat{A}(T\mathcal{M}_{\mu,k}) \wedge \exp c_1(\mathcal{L}) . \]

Note that this factor can be interpreted as the insertion of \( \exp \mu(S) \), which is the Donaldson map

\[ \mu : H_2(X) \rightarrow H^2(\mathcal{M}) , \]

evaluated at the co-dimension two locus \( S = \text{PD}(n) \in H_2(X) \).
We can also consider a Wilson loop at $x \in H_0(X)$.

$$W_{\mathcal{R}}(x) = Tr_{\mathcal{R}} \ P \exp \int_{S^1} (iA_5 + \sigma)dx^5.$$ 

This leads to the insertion of the Chern character of the universal bundle $\mathbb{E}$ in representation $\mathcal{R}$,

$$\text{Ch}_{\mathcal{R}}(\mathbb{E})/x,$$

where $/: H^\bullet(\mathcal{M}_{\mu,k} \times X) \times H_i(X) \to H^\bullet-i(\mathcal{M}).$
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To summarize,

\[
Z_{\mu}[\mathcal{R}, \{ x_{\mathcal{R}}, S \}] = \sum_{k=0}^{\infty}\mathcal{R}^{d(k)} \int_{\mathcal{M}_{\mu,k}} \hat{A}(TM_{\mu,k}) (\text{Ch}_{\mathcal{R}}(E)/x) \exp \mu(S) .
\]
Our approach to computation of the partition functions on $X \times S^1$: 

- The 5d $\mathcal{N} = 1$ theory on $S^1$ can be thought of as an effective 4d $\mathcal{N} = 2$ theory on $X$, with infinitely many Kaluza-Klein particles with mass $m_{\text{KK}} \sim n/R$, for $n \in \mathbb{Z}$.
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- Using the topological invariance on $X$, we scale the metric on $X$ as

$$g_{\mu\nu} \rightarrow t g_{\mu\nu}, \quad t \rightarrow \infty.$$

- The physics is replaced by a low energy effective theory on the Coulomb branch of 4d $\mathcal{N} = 2$ theory.

- The Coulomb branch is described by the Seiberg-Witten geometry.

- Computation of the partition function is reformulated into that of Seiberg-Witten invariants, with contributions from the $U$-plane integral.
Seiberg-Witten Geometry of Effective 4d $\mathcal{N} = 2$ Theory
Let us focus on $G = SU(2)$. Classical Coulomb branch of effective 4d theory is parametrized by

$$a = \frac{1}{R} \int_{S^1} (\sigma + iA_5) \, dx^5 \in \mathbb{R} \times S^1.$$  

The Coulomb branch effective theory is determined by the prepotential

$$F(a, \Lambda) = \frac{2}{R^2} \left( \text{Li}_3(e^{-R a}) - \zeta(3) \right) + a^2 \left( \log(\mathcal{R}) - \frac{\pi i}{2} \right) + \mathcal{O}(R\Lambda),$$

where $\Lambda$ is a dynamically generated scale of the 4d theory, which is related to the 5d gauge coupling

$$4 \log(\Lambda) = -\frac{8\pi^2 R}{g_{\text{YM}}^2}.$$

It is useful to define a dimensionless parameter $\mathcal{R} = R\Lambda$, which is the instanton counting parameter.
The order parameter for the quantum corrected Coulomb branch is the VEV of the fundamental Wilson loop,

\[ U = \langle \text{Tr}_F P \exp(Ra) \rangle = e^{Ra} + e^{-Ra} + O(R\Lambda) . \]

The Seiberg-Witten curve is parametrized by \( U \in \mathbb{C} \), \[\text{[Nekrasov 96]} \text{[Ganor, Morrison, Seiberg 96]} \text{[Eguchi, Sakai 02]}\]

\[ -R^2 X \left( \omega + \frac{1}{\omega} \right) = P(X)^2 , \quad P(X) = X^2 + UX + 1 . \]

with

\[ \lambda = \frac{1}{2\pi iR} \log(X) \frac{d\omega}{\omega} . \]

Then

\[ a = \int_A \lambda , \quad a_D = \int_B \lambda . \]
Seiberg-Witten geometry is an elliptic fibration over the U-plane. At generic value of $\mathcal{R}$, it is parametrized by $U \in \mathbb{C}$ with four singularities, where a BPS particle becomes massless.

$U_1$: A monopole becomes massless  \hspace{1cm}  U_2$: A dyon becomes massless \hspace{1cm}  U_3, U_4$: A dyonic instanton becomes massless

See also [Closset-Magureanu, 21]
From the Seiberg-Witten curve, we obtain the relation

\[ U(\tau) = \pm \left( -4R^2 \frac{\theta_2(\tau)^4 + \theta_3(\tau)^4}{\theta_2(\tau)^2 \theta_3(\tau)^2} + 4R^4 + 4 \right)^{1/2}. \]

- For \( R = 1 \), \( U(\tau) \) is modular invariant for \( \Gamma^0(8) \subset SL(2, \mathbb{Z}) \). Therefore, the fundamental domain is \( \mathcal{F}_{R=1} = \mathbb{H}/\Gamma^0(8) \).
- For generic \( R \), \( \exists \) branch points at \( U(\tau) = 0 \), whose position depends on \( R \).

Fundamental domain \( \mathcal{F}_R \), which is a double copy of \( \mathbb{H}/\Gamma^0(4) \) of 4d SYM.

See also [Aspman, Furrer, Manschot, 21]
U-plane integral
Following [Moore, Witten 97], the partition function of effective 4d theory for \( b_2^+(X) > 0 \) can be written as

\[
Z_{J, \mu}[\mathcal{R}, \mathbf{n}] = \Phi_{J, \mu}[\mathcal{R}, \mathbf{n}] + \sum_{i=1}^{4} Z_{J, \mu, i}^{SW}[\mathcal{R}, \mathbf{n}],
\]

where \( \Phi \) is the so-called “U-plane integral” contribution and \( Z^{SW} \) is the Seiberg-Witten contribution at the four singular points in the U-plane, where a BPS particle becomes massless.
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- For $b_2^+ > 1$, $Z_{J,\mu}$ is independent of metric on $X$.
- For $b_2^+ > 1$, $\Phi_{J,\mu}$ identically vanishes.
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where $\Phi$ is the so-called “U-plane integral” contribution and $Z^{SW}$ is the Seiberg-Witten contribution at the four singular points in the U-plane, where a BPS particle becomes massless.

- For $b_2^+ > 1$, $Z_{J,\mu}$ is independent of metric on $X$.
- For $b_2^+ > 1$, $\Phi_{J,\mu}$ identically vanishes.
- For $b_2^+ = 1$, the partition functions are expected to jump discontinuously as a function of metric on $X$.
- The metric dependence comes through $J$, the period point. $J \in H^2(X, \mathbb{R})$ with $J = *J$ and $J^2 = 1$.)
U-plane integral for $b_2^+ = 1$

\[ Z_{J,\mu} = \Phi_{J,\mu} + \sum_{i=1}^{4} Z_{J,\mu,i}^{SW} \]

- For $b_2^+ = 1$, $Z_{J,\mu}$ is a piecewise constant function of $J$. The dependence on $J$ only comes from the region $U \to \infty$.

- The $J$-dependence of $\Phi$ around the singularities at finite $U = U_i$ are also non-trivial, but they are canceled with the wall-crossing of $Z_{J,\mu,i}^{SW}$.

- We can utilize this fact to compute $Z_{J,\mu,i}^{SW}$ for $b_2^+ > 1$.

Let us focus on $b_2^+ = 1$ first.
Coulomb branch effective action

- The effective theory on Coulomb branch can be thought of as a $\mathcal{N} = 2 \ U(1)^G \times U(1)_I$ theory.

- We assume $b_1 = 0$ and $b_2^+ = 1$. Then only the zero mode contributes to the $U$-plane integral. [Moore-Witten 97] The Coulomb branch effective action restricted to zero modes reads

$$S = \int_X \frac{i}{16\pi} \left( \bar{\tau}_{ab} F_+^a \wedge F_+^b + \tau_{ab} F_-^a \wedge F_-^b \right) - \frac{1}{8\pi} \text{Im}(\tau_{ab}) D^a \wedge D^b$$

$$+ \frac{i\sqrt{2}}{16\pi} \bar{F}_{abc} \eta^a \chi^b \wedge (D + F_+)^c .$$

- The couplings $\tau_{ab}$ for $a = 1, 2$ are

$$\tau = -\frac{1}{2\pi i} \frac{\partial^2 F}{\partial a^2} , \quad \nu = -\frac{1}{2\pi i} \frac{\partial^2 F}{\partial a \partial m_I} , \quad \xi = -\frac{1}{2\pi i} \frac{\partial^2 F}{\partial m_I^2} ,$$

where $m_I = \frac{2}{R} \log(R)$. 
Integrating the zero modes, the U-plane integral becomes an integral over $\mathcal{F}_R$:

$$ \Phi_{J,\mu}[\mathcal{R}, n] = K_U \int_{\mathcal{F}_R} d\tau \wedge d\bar{\tau} \nu_R(\tau) C^n \psi^J_\mu(\tau, \bar{\tau}, v n/2, \bar{v} n/2) .$$

From the Seiberg-Witten geometry, the couplings can be written as $q$-series, where $q = e^{2\pi i \tau}$ [Göttsche, Nakajima, Yoshioka]

$$ \frac{\theta_1(\tau, v/2)}{\theta_4(\tau, v/2)} = -\mathcal{R} , \quad C = e^{2\pi i \xi} = \frac{\theta_4(\tau, v/2)}{\theta_4(\tau)} ,$$

and

$$ \psi^J_\mu(\tau, \bar{\tau}, z, \bar{z}) = e^{-2\pi y b^2} \sum_{k \in H^2(X, \mathbb{Z}) + \mu} \partial_{\bar{\tau}}(\sqrt{2} y B(k + b, J)) \cdot (-1)^{B(k, \mathcal{K}_X)} q^{-\frac{k_+^2}{2}} q^{k^2/2} - 2\pi i B(k_-, z) - 2\pi i B(k_+, \bar{z}) ,$$

with $b = \text{Im} \ z/y$. We defined $B(k_1, k_2) = \int_X k_1 \wedge k_2$. 
It is highly non-trivial that the integrand is single-valued under various monodromies in the U-plane.

A proper choice of $\tilde{Q}$-exact terms is crucial in establishing the single-valuedness.

This condition determines further quantization of the background flux $n$. 
Suppose that we have two period points $J$ and $J'$. Following [Korpas, Manschot 17] [Korpas, Manschot, Moore, Nidaiev 19], one can show

$$\Psi^J_{\mu}(\tau, \bar{\tau}, z, \bar{z}) - \Psi^{J'}_{\mu}(\tau, \bar{\tau}, z, \bar{z}) = \partial_{\bar{\tau}} \hat{\Theta}^J_{\mu, J'}(\tau, \bar{\tau}, z, \bar{z}),$$

$$\hat{\Theta}^J_{\mu, J'}(\tau, \bar{\tau}, z, \bar{z}) = \sum_{k \in H^2(X, \mathbb{Z}) + \mu} \frac{1}{2} \left[ E(\sqrt{2}yB(k + b, J)) - E(\sqrt{2}yB(k + b, J')) \right]$$

$$(-1)^{B(k, K_X)} q^{-k^2/2} e^{-2\pi iB(k, z)} ,$$

where $E(x) = \text{Erf}(\sqrt{\pi}x)$ and $y = \text{Im}(\tau)$, $b = \text{Im}(z)/y$. 
Suppose that we have two period points \( J \) and \( J' \). Following [Korpas, Manschot 17] [Korpas, Manschot, Moore, Nidaiev 19], one can show

\[
\Psi^J_\mu(\tau, \bar{\tau}, z, \bar{z}) - \Psi^{J'}_\mu(\tau, \bar{\tau}, z, \bar{z}) = \partial_{\bar{\tau}} \Theta^{J,J'}_\mu(\tau, \bar{\tau}, z, \bar{z}),
\]

\[
\Theta^{J,J'}_\mu(\tau, \bar{\tau}, z, \bar{z}) = \sum_{k \in H^2(X,\mathbb{Z}) + \mu} \frac{1}{2} \left[ E(\sqrt{2y}B(k + b, J)) - E(\sqrt{2y}B(k + b, J')) \right] (-1)^{B(k, K_X)} q^{-k^2/2} e^{-2\pi iB(k, z)},
\]

where \( E(x) = \text{Erf}(\sqrt{\pi}x) \) and \( y = \text{Im}(\tau), \ b = \text{Im}(z)/y \).

From this, we can write down the wall-crossing formula

\[
Z^J_\mu[\mathcal{R}, n] - Z^{J'}_\mu[\mathcal{R}, n] = \sum_{k \in \mathcal{W}_{J,J'}} 8 \left[ \nu_R(\tau) C^n^2 (-1)^{B(k, K)} q^{-k^2/2} e^{-2\pi iB(k, n v/2)} \right] q^0,
\]

where \( \mathcal{W}_{J,J'} = \{ k \mid B(k - n/4, J) > 0 \ \text{and} \ B(k - n/4, J') < 0 \} \).
There are two ways to evaluate this expression.

(1) Expand the integrand in small $R$ first and evaluate the $q^0$ term. This procedure reproduces the formula of [Göttsche, Nakajima, Yoshioka 06]. For example, $Z_\mu[n, R]$ for $X = \mathbb{P}^2$ can be computed via the WC formula, using a blowup and existence of vanishing chamber.

$$Z_0^{\mathbb{P}^2}[n, R] = \left\{ \begin{array}{ll}
R + R^5 + R^9 + \cdots & n = 0 \\
3R + 6R^5 + 10R^9 + \cdots & n = 1 \\
6R + 21R^5 + 56R^9 & n = 2 \\
10R + 56R^5 + 230R^9 + \cdots & n = 3 \\
\cdots & 
\end{array} \right.$$

which agrees with the interpretation

$$Z_0^{\mathbb{P}^2}[n, R] = \sum_{d \geq 0} \chi \left( \mathcal{M}_{0,d}^{\mathbb{P}^2}, \mathcal{O}(\mu(H^n)) \right) R^d.$$
However, in the five-dimensional theory point of view, it is more natural to

(2) Keep $\mathcal{R}$ finite and expand the integrand in small $q$ first, and take the $q^0$ term. Surprisingly, we find that the result does not agree with (1). In particular, for $b_2^+ = 1$, the result of the U-plane integral contains negative powers of $\mathcal{R}$.

A puzzle: How do we interpret the $\mathcal{R}^{-1}$ dependence of the partition function?
Toric localization
Toric localization

There is an alternative path integral derivation when $X$ is a smooth toric four-manifold.

Using the toric action, $Z$ can be written as a sum over contributions from the instantons localized at fixed loci on $X$. Schematically,

$$Z \sim \sum_k \int da \prod_{i=1}^{\chi} Z(a^i, \epsilon^i_1, \epsilon^i_2, R, \Lambda^i),$$

where $Z(a^i, \epsilon^i_1, \epsilon^i_2, R, \Lambda^i)$ is the K-theoretic Nekrasov partition function on $S^1 \times \mathbb{C}^2_{\epsilon^i_1 \epsilon^i_2}$, localized at $i$-th fixed locus on $X$.

This is the approach adopted by many authors including [Nekrasov] [GNY] [Hosseini et al.] [Crichigno et al.] [Bonelli et al.]...
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Remaining questions are

- What is the contour?
- Comparison with the U-plane integral approach?
Non-equivariant limit

Let us focus on the non-equivariant limit $\epsilon_1, \epsilon_2 \to 0$ of the partition function. The Nekrasov conjecture tells us that

$$\log Z(a, \epsilon_1, \epsilon_2, R, \Lambda) = \frac{1}{\epsilon_1 \epsilon_2} F(a, R, \Lambda) + \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} H(a, \Lambda)$$

$$+ \log A(a, \Lambda, R) + \frac{\epsilon_1^2 + \epsilon_2^2}{3\epsilon_1 \epsilon_2} \log B(a, \Lambda, R) + \cdots$$

Summing over contributions from all the fixed loci, we obtain the finite non-equivariant limit, the integrand becomes

$$Z_{k,\mu}(a, R, n) = \prod_{i=1}^{\chi} Z(a^i, \epsilon_1^i, \epsilon_2^i, R, \Lambda^i)$$

$$= \exp \left\{ \frac{1}{2} \frac{\partial^2 F}{\partial a^2} k^2 + \frac{R}{4} \frac{\partial^2 F}{\partial a \partial \log R} k n + \frac{R^2}{32} \frac{\partial^2 F}{\partial \log R^2} n^2 + \frac{\partial H}{\partial a} k KX + \chi A + \sigma B \right\} + O(\epsilon_1, \epsilon_2).$$
Contour encodes the metric dependence

Note that $b_2^+ = 1$ for a toric four-manifold. The partition function undergoes a wall-crossing.

The supersymmetric localization process gives

$$Z_{J,\mu}[\mathcal{R},n] = \sum_k \int \frac{dh}{h} \int_{\mathcal{M}} da \bar{a} \partial_{\bar{a}} Z_{k,\mu,\nu}(a,\bar{a},0,0,h)$$

$$= \sum_k \oint_{C_J} da \ Z_{k,\mu}(a) .$$

The contour $C_J$ depends on the choice of metric, $J$. After a careful analysis of the zero mode integrals, we arrive at the wall-crossing formula in this approach,

$$Z_{J,\mu}[\mathcal{R},n] - Z_{J',\mu}[\mathcal{R},n] = \sum_{k \in \mathcal{W}_{J,J'}} \left( \text{res}_{a=\infty} da + \text{res}_{a=-\infty} da \right) Z_{k,\mu}(a,\mathcal{R},n) ,$$

where $\mathcal{W}_{J,J'} = \{ k \mid B(k-n/4,J) > 0 \text{ and } B(k-n/4,J') < 0 \} . $

This agrees precisely with the WC formula in the U-plane integral approach.
$b_2^+(X) > 1$ and the Seiberg-Witten contribution
K-theoretic Donaldson invariants for $b_2^+ (X) > 1$

From

$$Z_{J, \mu} = \Phi_{J, \mu} + \sum_{i=1}^{4} Z_{J, \mu, i}^{SW},$$

and the fact that the wall-crossing at the strong coupling singularities cancels between $\Phi$ and $Z^{SW},$

$$\Phi_{J, \mu} |_{U_i} - \Phi_{J', \mu} |_{U_i} = - \left( Z_{J, \mu, i}^{SW} - Z_{J', \mu, i}^{SW} \right),$$

it is possible to derive the K-theoretic Donaldson invariants for $b_2^+ (X) > 1.$ We find

$$Z_\mu [\mathcal{R}, n] = \sum_{l \mod 4} e^{-i \pi (3 \chi_h / 2 + 2 \mu^2) l} G_\mu (e^{-i \pi l / 2} \mathcal{R}, n),$$

where

$$G_\mu (\mathcal{R}, n) = \frac{2^{2 \chi + 3 \sigma - \chi_h}}{(1 - \mathcal{R}^2)^{n^2 / 2 + \chi_h}} \sum_c \text{SW}(c) \left( \frac{1 - \mathcal{R}}{1 + \mathcal{R}} \right)^{B(n, c) / 2} e^{\pi i B(K + c, \mu)}.$$

This reproduces the result of [Göttsche, Kool, Williams 19].
Conclusion
The topological correlator of 5d $\mathcal{N} = 1$ $SU(2)$ gauge theory computes the K-theoretic Donaldson invariant

$$Z_{J,\mu}[\mathcal{R}, \{x, S\}] = \sum_k \mathcal{R}^{d(k)} \int_{\mathcal{M}_{k,\mu}} \hat{A}_R(TM_{k,\mu}) (Ch_R(E)/x) e^{\mu(S)}.$$ 

In particular, we derived the metric dependence of $Z_{J,\mu}$ for $b_1 = 0$ and $b_2^+ = 1$ in the perspective of the U-plane integral, and also in the $SU(2)$ gauge theory perspective. We show that they agree in a non-trivial way.

Using the U-plane integral approach, the computation can be generalized to $b_2^+ > 1$. This reproduces the result of [Göttsche, Kool, Williams 19]

The appearance of negative powers of $\mathcal{R}$ for $b_2^+(X) = 1$ remains to be understood.
Future directions

- Extension to general rank 1 $E_N$ theories and 5d $\mathcal{N} = 1^*$ theory. The latter is expected to compute the $\chi_y$-genus of the moduli space, which is identified with the K-theoretic Vafa-Witten invariants.

- Using the U(1) Kaluza-Klein symmetry, it is possible to compute the partition functions on more general five-manifolds (e.g., $S^5$) via the U-plane integral approach.

- Uplift to 6d $\mathcal{N} = (1, 0)$ gauge theories. It is expected to compute the “elliptic invariants” on the moduli space of instantons.

- Extension to $b_1(X) > 0$ and dimensional reduction to 3d $\mathcal{N} = 4$ theories.

- More general topological defects.