Topological Langlands duality via skeins

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Gunningham

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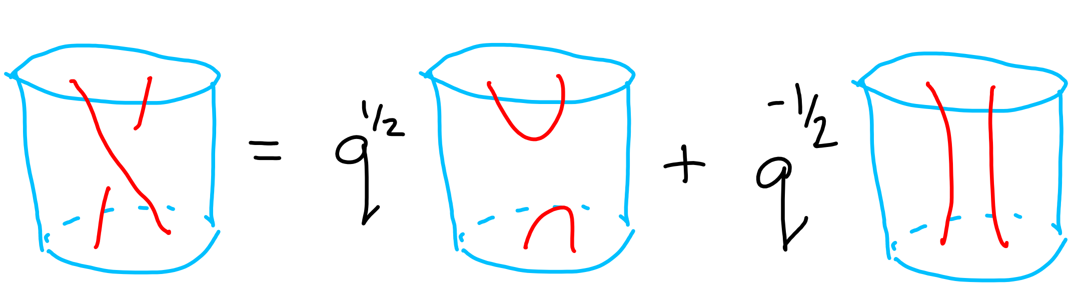
Let M be an oriented 3-manifold, and G a reductive group. Fix q in $\mathbb{C}^\times$.

The skein module $Sk_G(M)$ is the span over $\mathbb{C}$ of G-colored ribbon graphs in M, modulo the skein relations.

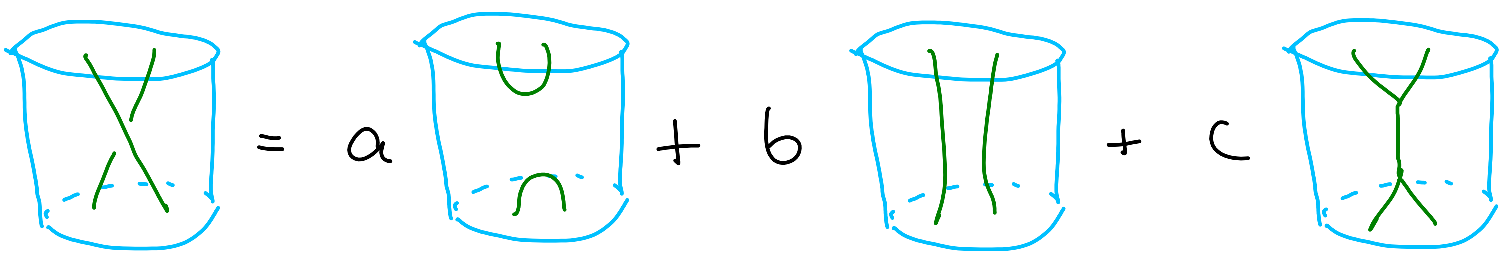
$= \mathbb{C}^2 = V(1)$

$= \mathbb{C}^3 = V(2)$

SL_2:



PGL_2:



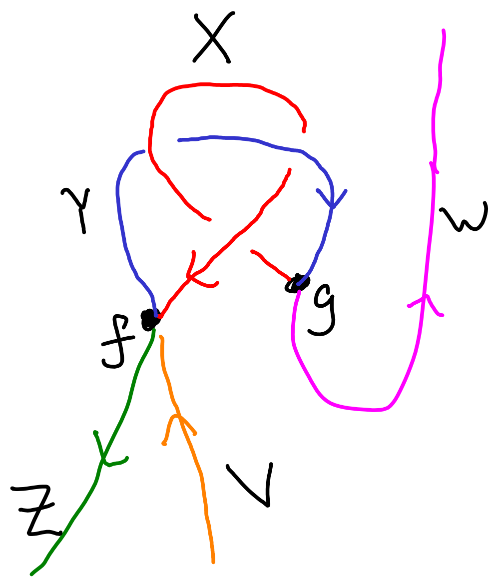
"A two-dimensional vector space has the marvelous property that any three vectors satisfy a relation of linear dependence."

E. Witten

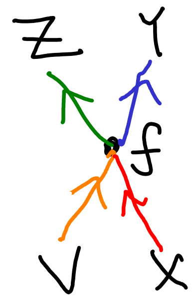
Skein modules for a general G:

A typical G-ribbon graph looks like this:

Finite dimensional U_q-modules w/ correct K weights.

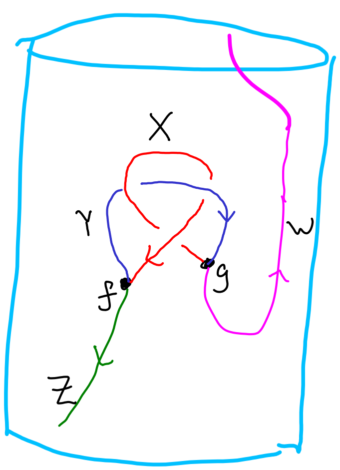


$X, Y, Z, V, W \in \mathrm{Rep}_q G$



$\in \mathrm{Hom}(V \otimes X, Z \otimes Y)$

Skein relations are the (local) kernel of Reshetikhin–Turaev evaluation.



$\in \mathrm{Hom}(Z^*, W)$

(Reshetikhin–Turaev, Walker 90's)

Basic facts about skein modules:

1) At $q=1$, we have: $Sk_G(M) \cong \mathcal{O}(Loc_G^{Betti}(M))$

2) For $q \neq 1$, we have that $Sk_G(\Sigma \times I)$ is a deformation-quantization of the Atiyah–Bott–Goldman Poisson bracket.

3) However, for generic q, and for M closed, we have:

$$\dim_{\mathbb{C}}(Sk_G(M)) < \infty$$

Conjectured by Witten ~'16, proved in GJS19.

4) We have a $(3,2)$-dimensional TQFT $Sk_G : Cob_{2,3} \longrightarrow Cat_{\mathbb{C}}$

$$M^3 \longmapsto Sk_G(M^3) \in Vect_{\mathbb{C}} \qquad \partial M^3 = \emptyset$$

$$\Sigma \longmapsto Sk_G(\Sigma) \in Cat_{\mathbb{C}}$$

$$M^3 \longmapsto SkFun(M) : \overbrace{Sk_G}(\Sigma_{in}) \longrightarrow \overbrace{Sk_G}(\Sigma_{out})$$

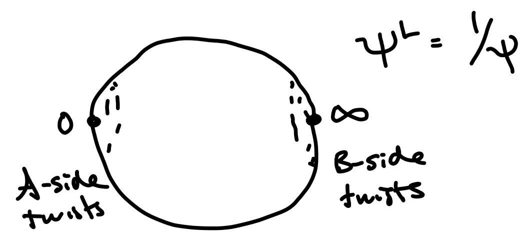
Langlands duality for skein modules

- Expectation: $Sk_G(M)$ models a distinguished sector of $\mathcal{H}^{\psi}_G(M)$, the space of states for Kapustin–Witten's twist of N=4 4D SYM. Here, we have: $q = e^{\pi i \cdot \frac{1}{\psi}}$, $\psi \in \mathbb{CP}^1$

- Let G be a simple group and let G^L be its Langlands dual group.

- Conjecture: we have an equality of integer dimensions:

$$\boxed{\dim_{\mathbb{C}} Sk_{G^L}(M) = \dim_{\mathbb{C}} Sk_G(M)}$$



Relatives/antecedents:

Beilinson–Drinfeld 00's, Arinkin–Gaitsgory '10s: $\mathrm{IndCoh}(Loc^{dR}_{G^L}(\Sigma)) \simeq \mathcal{D}\text{-mod}(Bun_G(\Sigma))$

Frenkel, Gaitsgory, Kapustin–Witten: $\mathcal{D}\text{-mod}^{\psi}(Bun_{G^L}(\Sigma)) \simeq \mathcal{D}\text{-mod}^{1/\psi}(Bun_G(\Sigma))$

Ben-Zvi–Nadler: $\mathrm{IndCoh}(Loc^{Betti}_{G^L}(\Sigma)) \simeq Sh(Bun_G(\Sigma))$

Mazur, Kapranov, Reznikov; Kim: Arithmetic topology: number fields as 3-manifolds

Numerical evidence:

Only very recently have <u>any</u> skein module dimensions been computed.

	$G = SL_2$	$G^{L} = PGL_2$
$\Sigma_g \times S^1$	$2^{2g+1} + 2g - 1$ Carrega, Gilmer, Masbaum Detcherry, Wolff '19	$2^{2g+1} + 2g - 1$ Gunningham–J–Safronov via 1-form symmetries '22
	$G = SL_N$	$G^{L} = PGL_N$
T^3	$(P * \mu * \text{cube})(N)$ $= \sum_{N = d \cdot e \cdot f} P(d) \cdot \mu(e) \cdot f^3$ Gunningham–J–Vazirani–Yang via Schur–Weyl duality, '22	$(P * \mu * \text{cube})(N)$ $= \sum_{N = d \cdot e \cdot f} P(d) \cdot \mu(e) \cdot f^3$

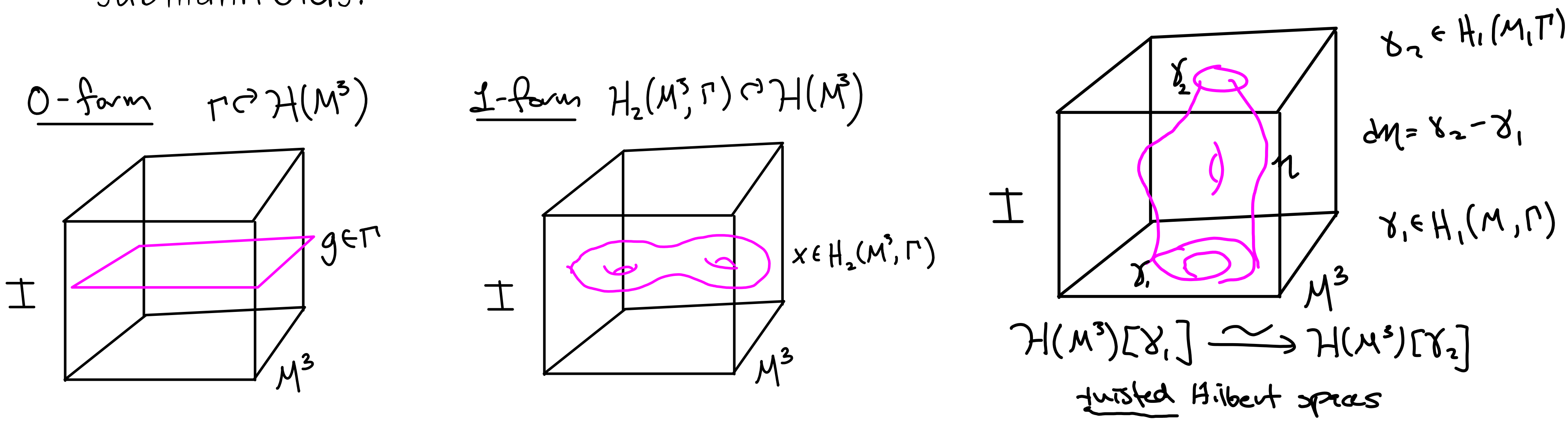
Today

$P(d)$ = Partition number = # Young diagrams w/ d boxes

$\mu(e)$ = Mobius inversion function

Appearance of 1-form symmetries:

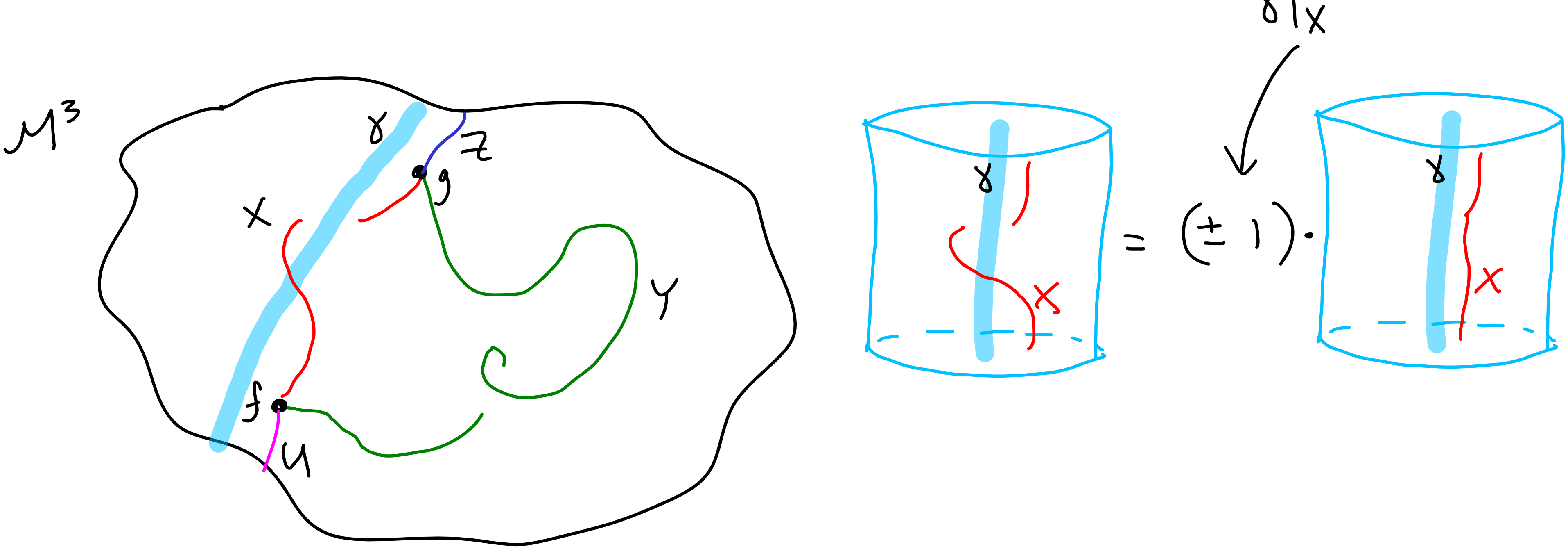
- Note that PGL_N is also a quotient, $PGL_N = SL_N / Z(SL_N)$.
- This induces an action of $\underset{\underset{\text{?}}{\text{II}}}{Z(SL_N)}$ on Sk_{SL_N} by 1-form symmetries.
- And induces an action of $\pi_1(PGL_N)$ on Sk_{PGL_N} by 1-form symmetries.
- While ordinary Γ-symmetries of a QFT allow insertion along codimension-one submanifolds...
- 1-form symmetries of a QFT allow insertion along codimension-two submanifolds:



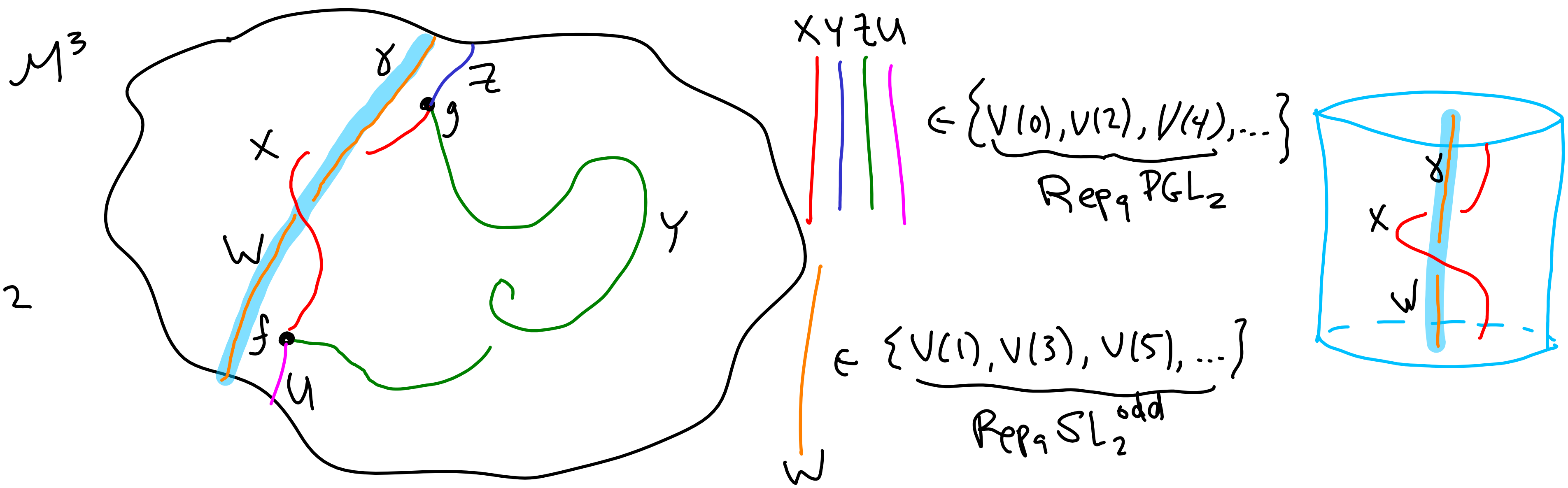
Idea: compute PGL_N-skein modules as twists of SL_N-skein modules.

Skein description of twists by 1-form symmetry:

SL_2

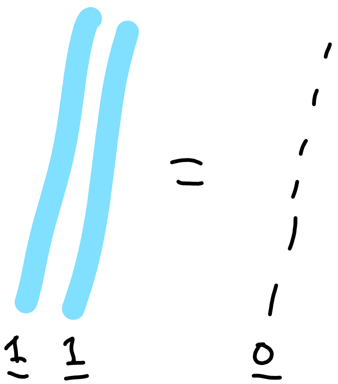


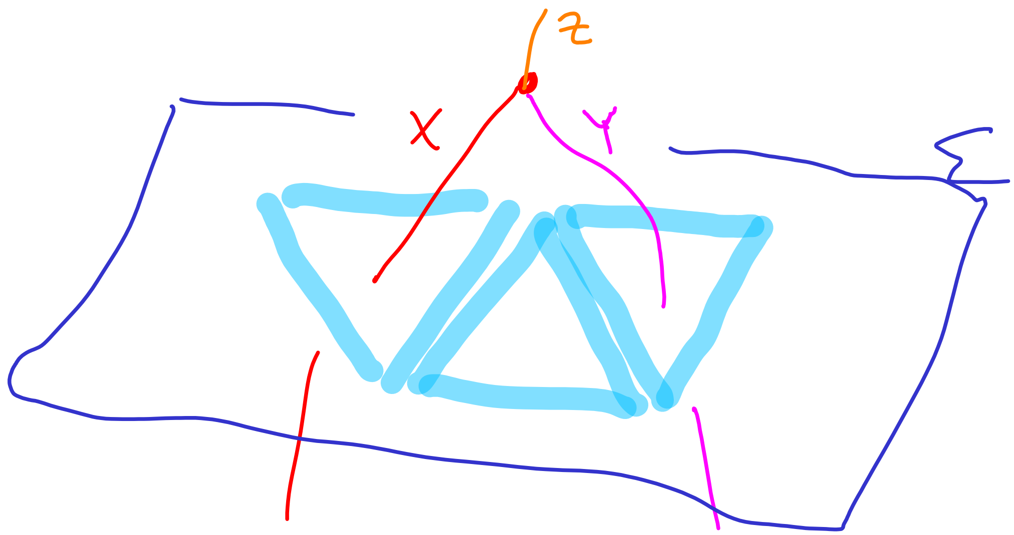
PGL_2



Skein description of action/grading by 1-form symmetry:

SL_2: $\quad Z = \mathbb{Z}/2 = \{\underline{0}, \underline{1}\}$.





$= \mathbb{Z}/2$-intersection pairing

Action

$$H_2(M,Z) \curvearrowright \mathcal{H} \quad \leftrightsquigarrow \quad \mathcal{H} = \bigoplus_{\substack{a \in H_2(M,Z)^\vee \\ = H_1(M,Z^\vee)}} \mathcal{H}_a$$

Bi-graded skein modules

- The center 1-form symmetry induces twisted skein modules, and each twisted skein module has a remaining grading:

$$\mathrm{Sk}^{a,b}_{G^{sc}}(M) \in \mathrm{Vect}_{\mathbb{C}} \quad \text{w/} \quad \begin{array}{l} a \in H_1(M, Z(G^{sc})^\vee) \ (\text{grading}) \\ b \in H_1(M, Z(G^{sc})) \ (\text{twist}) \end{array}$$

$$\mathrm{Sk}_{G^{sc}}(M) = \bigoplus_a \mathrm{Sk}^{a,0}_{G^{sc}}(M)$$

- The fundamental group 1-form symmetry induces twisted skein modules, and each twisted skein modules has a remaining grading:

$$\mathrm{Sk}^{a,b}_{G^{ad}}(M) \in \mathrm{Vect}_{\mathbb{C}} \quad \text{w/} \quad \begin{array}{l} a \in H_1(M, \pi_1(G^\vee)) \ (\text{twist}) \\ b \in H_1(M, \pi_1(G^{ad})) \ (\text{grading}) \end{array}$$

$$\mathrm{Sk}_{G^{ad}}(M) = \bigoplus_b \mathrm{Sk}^{0,b}_{G^{ad}}(M)$$

Theorem (GJS): We have natural isomorphisms:

$$\mathrm{Sk}^{a,b}_{G^{sc}}(M) \cong \mathrm{Sk}^{a,b}_{G^{ad}}(M)$$

Back to Langlands duality:

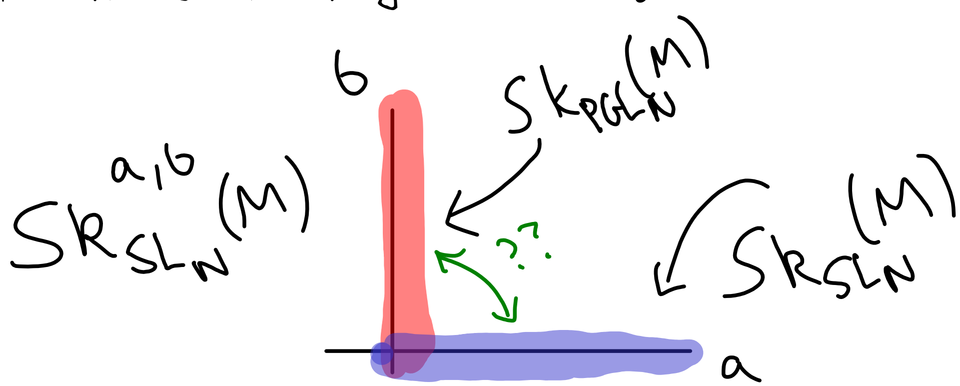
Theorem (GJS): We have natural isomorphisms:

$$\mathrm{Sk}^{a,b}_{G^{sc}}(M) \cong \mathrm{Sk}^{a,b}_{G^{ad}}(M)$$

However, recall that S-/Langlands duality should exchange electric and magnetic 1-form symmetries, so it predicts instead:

$$\mathrm{Sk}^{a,b}_{SL_N}(M) \cong \mathrm{Sk}^{b,a}_{PGL_N}(M) \quad \left(\cong \mathrm{Sk}^{b,a}_{SL_N}(M)\right)$$

Hence, to prove Langlands duality in some case, we need to compute ordinary G-skein modules in all degrees, plus all twisted skein modules in degree zero.



Twisted character varieties at $q=1$

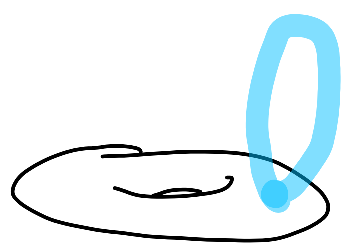
Recall that at $q=1$ we have $Sk_{G^{sc}}(M) = \mathcal{O}(Loc_G(M))$

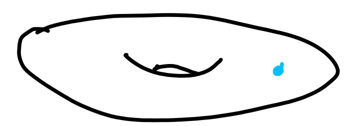
Likewise, at $q=1$ we have $Sk^{*,b}_{G^{sc}}(M) = \mathcal{O}(Loc^{tw,b}_{G^{sc}}(M)) \circlearrowleft H_2(M,\mathbb{Z})$

A small but well-known miracle occurs: twisted character varieties are (often) smooth!

e.g.

$Sk^{*,m}_{SL_N}(T^3)$:





$$Loc^{tw}_G(T_2) = \{(A,B) \in G \text{ s.t. } ABA^{-1}B^{-1} = \zeta_m\}$$

$$\Rightarrow \quad Sk^{*,b}_{G^{s.c}}(M) \cong H_0(Loc^{tw,b}_{G^{sc}}(M))$$

Summary:

1) We expect skein modules compute a "piece" of the Kapustin–Witten twist at generic parameters.

2) This would suggest a Langlands duality between skein theories for G and its Langlands dual.

3) We don't check this directly (we do not even propose an isomorphism!), instead we compute dimensions independently.

4) For this we establish a natural compatibility with electric–magnetic charges (1–form symmetries) and conjecture compatibility with Langlands duality. We confirm this in some cases.

Further directions:

5) Intrinsic geometric description of Hilbert space via "categorified Donaldson–Thomas invariants (Gunningham–Safronov)
6) Value of A–model at $\Psi=0$, precise form of classical Langlands duality for 3–manifolds??