Non-rational $\hat{su}(2)$ cosets and Liouville field theory

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Results obtained in collaboration with Błażej Ruba

String Math 2022, University of Warsaw, Poland.

July 11-15, 2022
Lemma

A tensor product of the highest weight representations of the affine $\hat{\mathfrak{sl}}_2$ algebra: the Verma module $\mathcal{V}^{k,j}$ and $\mathcal{H}^{1,\epsilon} = \mathcal{V}^{1,\epsilon}/\text{rad } \mathcal{V}^{1,\epsilon}$ with $\epsilon = 0, \frac{1}{2}$, decomposes as:

$$\mathcal{V}^{k,j} \otimes \mathcal{H}^{1,\epsilon} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{V}^{k+1,j_n} \otimes \mathcal{V}^{c,\Delta_n},$$

where $\mathcal{V}^{c,\Delta_n}$ is the Virasoro Verma module.
\( \mathfrak{sl}_2 \) is a Lie algebra with basis \( \{ J^\pm_n, J^0_n \}_{n \in \mathbb{Z}} \cup \{ K \} \) and nonzero commutators

\[
\begin{align*}
[J^0_n, J^\pm_m] &= \pm J^\pm_{n+m}, & [J^+_n, J^-_m] &= 2J^0_{n+m} + n\delta_{n+m}K & [J^0_n, J^0_m] &= \frac{n}{2}\delta_{n+m}K.
\end{align*}
\]
\( \overset{\sim}{\mathfrak{l}}_2 \) is a Lie algebra with basis \( \{ J_n^{\pm}, J_n^0 \} \in \mathbb{Z} \cup \{ K \} \) and nonzero commutators

\[
\begin{align*}
[ J_n^0, J_m^{\pm} ] &= \pm J_{n+m}^{\pm}, \\
[ J_n^+, J_m^- ] &= 2J_{n+m}^0 \pm n\delta_{n+m}K \\
[ J_n^0, J_m^0 ] &= \frac{n}{2} \delta_{n+m}K.
\end{align*}
\]

Let \( \overset{\sim}{\mathfrak{l}}_2^+ \) be the subalgebra of \( \overset{\sim}{\mathfrak{l}}_2 \) generated by \( J_0^+ \) and \( \{ J_n^a \} \in \mathbb{Z} \cup \{ K - k, J_0^0 - j \} \). The Verma module \( \mathcal{V}^{k,j} \) is the quotient of the universal enveloping algebra of \( \overset{\sim}{\mathfrak{l}}_2 \) by the left ideal generated by \( \overset{\sim}{\mathfrak{l}}_2^+ \cup \{ K - k, J_0^0 - j \} \). The image of 1 in \( \mathcal{V}^{k,j} \) is denoted by \( \nu_{k,j} \).
\( \widehat{\mathfrak{sl}}_2 \) is a Lie algebra with basis \( \{ J^\pm_n, J_0^n \}_{n \in \mathbb{Z}} \cup \{ K \} \) and nonzero commutators

\[
\begin{align*}
[J_n^0, J_m^\pm] &= \pm J_{n+m}^\pm, \\
[J_n^+, J_m^-] &= 2J_{n+m}^0 + n\delta_{n+m}K \\
[J_n^0, J_m^0] &= \frac{n}{2}\delta_{n+m}K.
\end{align*}
\]

Let \( \widehat{\mathfrak{sl}}_2^+ \) be the subalgebra of \( \widehat{\mathfrak{sl}}_2 \) generated by \( J_0^+ \) and \( \{ J_n^a \}_{n \geq 1} \).

Verma module \( \mathcal{V}^{k,j} \) is the quotient of the universal enveloping algebra of \( \widehat{\mathfrak{sl}}_2 \) by the left ideal generated by \( \widehat{\mathfrak{sl}}_2^+ \cup \{ K - k, J_0^0 - j \} \). The image of \( 1 \) in \( \mathcal{V}^{k,j} \) is denoted by \( v_{k,j} \).

Every irreducible, highest weight module over \( \widehat{\mathfrak{sl}}_2 \) is isomorphic to some (unique) \( \mathcal{H}^{k,j} = \mathcal{V}^{k,j}/\text{rad}(\mathcal{V}^{k,j}) \), where \( \text{rad}(\mathcal{V}^{k,j}) \) is the union of all proper submodules of \( \mathcal{V}^{k,j} \).
\( \hat{\mathfrak{sl}}_2 \) is a Lie algebra with basis \( \{ J^\pm_n, J^0_n \}_{n \in \mathbb{Z}} \cup \{ K \} \) and nonzero commutators:

\[
[J^0_n, J^\pm_m] = \pm J^\pm_{n+m}, \quad [J^+_n, J^-_m] = 2J^0_{n+m} + n\delta_{n+m}K \quad [J^0_n, J^0_m] = \frac{n}{2}\delta_{n+m}K.
\]

Let \( \hat{\mathfrak{sl}}_2^+ \) be the subalgebra of \( \hat{\mathfrak{sl}}_2 \) generated by \( J^+_0 \) and \( \{ J^a_n \}_{n \geq 1} \).

Verma module \( \mathcal{V}^{k,j} \) is the quotient of the universal enveloping algebra of \( \hat{\mathfrak{sl}}_2 \) by the left ideal generated by \( \hat{\mathfrak{sl}}_2^+ \cup \{ K - k, J^0_0 - j \} \). The image of \( 1 \) in \( \mathcal{V}^{k,j} \) is denoted by \( \nu^{k,j} \).

Every irreducible, highest weight module over \( \hat{\mathfrak{sl}}_2 \) is isomorphic to some (unique) \( \mathcal{H}^{k,j} = \mathcal{V}^{k,j} / \text{rad}(\mathcal{V}^{k,j}) \), where \( \text{rad}(\mathcal{V}^{k,j}) \) is the union of all proper submodules of \( \mathcal{V}^{k,j} \).

This explains \( \mathcal{V}^{k,j} \otimes \mathcal{H}^{1,\epsilon} \).
With the help of the local currents $J^a(z) = \sum_{n \in \mathbb{Z}} \frac{j_n^a}{z^{n+1}}$, one defines for $k \neq -2$ the Sugawara field:

$$T(z) = \frac{1}{k+2} \left( : J^0(z) J^0(z) : + \frac{1}{2} : J^+(z) J^-(z) : + \frac{1}{2} : J^-(z) J^+(z) : \right).$$
With the help of the local currents $J^a(z) = \sum_{n \in \mathbb{Z}} \frac{j_n^a}{z^{n+1}}$, one defines for $k \neq -2$ the Sugawara field:

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Its modes, defined as $T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}$ generate the Virasoro algebra $\text{Vir}$ with the central charge $c_k = \frac{3k}{k+2}$.
Withe the help of the local currents \( J^a(z) = \sum_{n \in \mathbb{Z}} \frac{j_n^a}{z^{n+1}} \), one defines for \( k \neq -2 \) the Sugawara field:

\[
T(z) = \frac{1}{k+2} \left( : J^0(z) J^0(z) : + \frac{1}{2} : J^+(z) J^-(z) : + \frac{1}{2} : J^-(z) J^+(z) : \right).
\]

Its modes, defined as \( T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}} \) generate the Virasoro algebra \( \text{Vir} \) with the central charge \( c_k = \frac{3k}{k+2} \).

Let \( \text{Vir}^+ \) be the subalgebra of \( \text{Vir} \) generated by \( \{L_n\}_{n \geq 1} \).

Verma module \( V^{c,\Delta} \) is the quotient of the universal enveloping algebra of \( \text{Vir} \) by the left ideal generated by \( \text{Vir}^+ \cup \{L_0 - \Delta\} \).
With the help of the local currents $J^a(z) = \sum_{n \in \mathbb{Z}} \frac{j^a_n}{z^{n+1}}$, one defines for $k \neq -2$ the Sugawara field:

$$T(z) = \frac{1}{k+2} \left( : J^0(z) J^0(z) : + \frac{1}{2} : J^+(z) J^-(z) : + \frac{1}{2} : J^-(z) J^+(z) : \right).$$

Its modes, defined as $T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}$, generate the Virasoro algebra Vir with the central charge $c_k = \frac{3k}{k+2}$.

Let Vir$^+$ be the subalgebra of Vir generated by $\{L_n\}_{n \geq 1}$.

Verma module $V^{c,\Delta}$ is the quotient of the universal enveloping algebra of Vir by the left ideal generated by Vir$^+ \cup \{L_0 - \Delta\}$.

This (partially) explains $V^{k+1,j_n} \otimes V^{c,\Delta_n}$ in

$$V^{k,j} \otimes \mathcal{H}^{1,\epsilon} \cong \bigoplus_{n \in \mathbb{Z}} V^{k+1,j_n} \otimes V^{c,\Delta_n},$$
Let $\mathfrak{f}$ be the Lie superalgebra spanned by odd $\{\psi_n^i, \bar{\psi}_n^i\}_{n \in \mathbb{Z} + \frac{1}{2}}$ and even $I$, with only nonzero superbrackets

$$[\psi_n^i, \bar{\psi}_m^j] = \delta_{n+m,0} \delta^{i,j} I.$$ 

The Fock space $\mathcal{F}$ is defined as the quotient of $U\mathfrak{f}$ by the left ideal generated by $\{\psi_n^i, \bar{\psi}_n^i\}_{n > 0} \cup \{I - 1\}$. $f_0$ is the image of $1$ in $\mathcal{F}$ and $f_{\frac{1}{2}} = \bar{\psi}_{-\frac{1}{2}}^1 f_0$. 
Let $\mathfrak{f}$ be the Lie superalgebra spanned by odd $\{\psi^i_n, \overline{\psi}^i_n\}_{n \in \mathbb{Z} + \frac{1}{2}}$ and even $I$, with only nonzero superbrackets

$$[\psi^i_n, \overline{\psi}^j_m] = \delta_{n+m,0}\delta^{i,j}I.$$

The Fock space $\mathcal{F}$ is defined as the quotient of $U\mathfrak{f}$ by the left ideal generated by $\{\psi^i_n, \overline{\psi}^i_n\}_{n > 0} \cup \{I - 1\}$. $f_0$ is the image of 1 in $\mathcal{F}$ and $f_{1/2} = \overline{\psi}^{-1/2}f_0$.

$\mathcal{F}$ is acted upon by local fields $\psi^i(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\psi^i_n}{z^{n+\frac{1}{2}}}$, $\overline{\psi}^i(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\overline{\psi}^i_n}{z^{n+\frac{1}{2}}}$.

Modes of the currents:

$$K^a(z) = \sum_{i,j=1}^{2} \tau_{i,j} : \overline{\psi}^i(z)\psi^j(z) :,$$

where $\tau^0 = \frac{1}{2}\sigma^3$, $\tau^\pm = \frac{1}{2} (\sigma^1 \pm i\sigma^2)$, satisfy the $\widehat{sl}_2$ algebra with $k = 1$. 
Let $\mathfrak{f}$ be the Lie superalgebra spanned by odd $\{\psi_n, \overline{\psi}_n\}_{n \in \mathbb{Z} + \frac{1}{2}}$ and even $I$, with only nonzero superbrackets

\[ [\psi_n, \overline{\psi}_m] = \delta_{n+m,0} \delta^{i,j} I. \]

The Fock space $\mathcal{F}$ is defined as the quotient of $U\mathfrak{f}$ by the left ideal generated by $\{\psi_n, \overline{\psi}_n\}_{n > 0} \cup \{I - 1\}$. $f_0$ is the image of $1$ in $\mathcal{F}$ and $f_{\frac{1}{2}} = \overline{\psi}_{-\frac{1}{2}} f_0$.

$\mathcal{F}$ is acted upon by local fields $\psi^i(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\psi_n^i}{z^{n+\frac{1}{2}}}$, $\overline{\psi}^i(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\overline{\psi}_n^i}{z^{n+\frac{1}{2}}}$.

Modes of the currents:

\[ K^a(z) = \sum_{i,j=1}^{2} \tau^a_{i,j} : \overline{\psi}^i(z) \psi^j(z) :, \]

where $\tau^0 = \frac{1}{2} \sigma^3$, $\tau^\pm = \frac{1}{2} (\sigma^1 \pm i \sigma^2)$, satisfy the $\widehat{sl}_2$ algebra with $k = 1$. 
Let $\mathfrak{f}$ be the Lie superalgebra spanned by odd $\{\psi^i_n, \overline{\psi}^i_n\}_{n \in \mathbb{Z} + \frac{1}{2}}$ and even $I$, with only nonzero superbrackets

$$[\psi^i_n, \overline{\psi}^j_m] = \delta_{n+m,0} \delta^{i,j} I.$$

The Fock space $\mathcal{F}$ is defined as the quotient of $\mathbb{U}\mathfrak{f}$ by the left ideal generated by $\{\psi^i_n, \overline{\psi}^i_n\}_{n > 0} \cup \{I - 1\}$. $f_0$ is the image of 1 in $\mathcal{F}$ and $f_{\frac{1}{2}} = \overline{\psi}^1_{-\frac{1}{2}} f_0$.

$\mathcal{F}$ is acted upon by local fields $\psi^i(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\psi^i_n}{z^{n+\frac{1}{2}}}$, $\overline{\psi}^i(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\overline{\psi}^i_n}{z^{n+\frac{1}{2}}}$.

Modes of the currents:

$$K^a(z) = \sum_{i,j=1}^{2} \tau^a_{i,j} : \overline{\psi}^i(z) \psi^j(z) :,$$

where $\tau^0 = \frac{1}{2} \sigma^3$, $\tau^\pm = \frac{1}{2} (\sigma^1 \pm i \sigma^2)$, satisfy the $\hat{\mathfrak{sl}}_2$ algebra with $k = 1$.

Proposition

- $\hat{\mathfrak{sl}}_2$-submodule of $\mathcal{F}$ generated by $f_\epsilon$ is isomorphic to $\mathcal{H}^{1,\epsilon}$. 

Leszek Hadasz  
Non-rational $\hat{\mathfrak{su}}(2)$ cosets and Liouville field theory
Fix $\kappa \neq 0$ and let $\mathfrak{w}$ be the Lie algebra spanned by $a_n, \beta_n, \gamma_n, n \in \mathbb{Z}$, with commutators

$$[\gamma_m, \beta_n] = \delta_{m+n}l, \quad [a_m, a_n] = \frac{m}{2} \delta_{m+n}l.$$
Free fields: Wakimoto representation

- Fix $\kappa \neq 0$ and let $\mathfrak{w}$ be the Lie algebra spanned by $a_n, \beta_n, \gamma_n, \ n \in \mathbb{Z}$, with commutators

$$[\gamma_m, \beta_n] = \delta_{m+n}l, \quad [a_m, a_n] = \frac{m}{2} \delta_{m+n}l.$$ 

- We define a $\mathfrak{w}$-module $\mathcal{W}^{\kappa,j}$ as the quotient of $U\mathfrak{w}$ by the left ideal generated by $\{a_0 - \kappa^{-1}j, l - 1\} \cup \{\beta_n, \gamma_{n+1}, a_{n+1}\}_{n=0}^{\infty}$. $\mathcal{W}^{\kappa,j}$ is an irreducible representation of $\mathfrak{w}$. Let $\mathfrak{w}_{\kappa,j}$ be the image of 1 in $\mathcal{W}^{\kappa,j}$. 

Leszek Hadasz

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$$[\gamma_m, \beta_n] = \delta_{m+n}I, \quad [a_m, a_n] = \frac{m}{2} \delta_{m+n}I.$$ 

We define a $\mathfrak{w}$-module $\mathcal{W}^{\kappa,j}$ as the quotient of $U\mathfrak{w}$ by the left ideal generated by $\{a_0 - \kappa^{-1}j, I - 1\} \cup \{\beta_n, \gamma_{n+1}, a_{n+1}\}_{n=0}^\infty$. $\mathcal{W}^{\kappa,j}$ is an irreducible representation of $\mathfrak{w}$. Let $\mathfrak{w}_{\kappa,j}$ be the image of $1$ in $\mathcal{W}^{\kappa,j}$.

Using

$$\partial \phi(z) = \sum_{n \in \mathbb{Z}} \frac{a_n}{z^{n+1}}, \quad \beta(z) = \sum_{n \in \mathbb{Z}} \frac{\beta_n}{z^{n+1}}, \quad \gamma(z) = \sum_{n \in \mathbb{Z}} \frac{\gamma_n}{z^n}.$$ 

we define the $\hat{\mathfrak{sl}}_2$ currents with $k = \kappa^2 - 2$:

$$J^+(z) = \beta(z), \quad J^0(z) = :\gamma(z)\beta(z): + \kappa \partial \phi(z),$$
$$J^-(z) = - :\gamma(z)^2\beta(z): - 2\kappa \gamma(z) \partial \phi(z) - k \partial \gamma(z).$$
Fix $\kappa \neq 0$ and let $\mathfrak{w}$ be the Lie algebra spanned by $a_n, \beta_n, \gamma_n, \ n \in \mathbb{Z}$, with commutators

$$[\gamma_m, \beta_n] = \delta_{m+n}I, \quad [a_m, a_n] = \frac{m}{2} \delta_{m+n}I.$$ 

We define a $\mathfrak{w}$-module $\mathcal{V}_{\kappa, j}$ as the quotient of $U\mathfrak{w}$ by the left ideal generated by $\{a_0 - \kappa^{-1}j, I - 1\} \cup \{\beta_n, \gamma_{n+1}, a_{n+1}\}_{n=0}^{\infty}$. $\mathcal{V}_{\kappa, j}$ is an irreducible representation of $\mathfrak{w}$. Let $\mathfrak{w}_{\kappa, j}$ be the image of 1 in $\mathcal{V}_{\kappa, j}$.

Using

$$\partial \phi(z) = \sum_{n \in \mathbb{Z}} \frac{a_n}{z^{n+1}}, \quad \beta(z) = \sum_{n \in \mathbb{Z}} \frac{\beta_n}{z^{n+1}}, \quad \gamma(z) = \sum_{n \in \mathbb{Z}} \frac{\gamma_n}{z^n}.$$ 

we define the $\widehat{sl}_2$ currents with $k = \kappa^2 - 2$:

$$J^+(z) = \beta(z), \quad J^0(z) =: \gamma(z) \beta(z) : + \kappa \partial \phi(z),$$

$$J^-(z) = - : \gamma(z)^2 \beta(z) : - 2 \kappa \gamma(z) \partial \phi(z) - k \partial \gamma(z).$$
Fix $\kappa \neq 0$ and let $\mathfrak{w}$ be the Lie algebra spanned by $a_n, \beta_n, \gamma_n$, $n \in \mathbb{Z}$, with commutators

$$[\gamma_m, \beta_n] = \delta_{m+n} l, \quad [a_m, a_n] = \frac{m}{2} \delta_{m+n} l.$$ 

We define a $\mathfrak{w}$-module $\mathcal{W}^{\kappa, j}$ as the quotient of $U\mathfrak{w}$ by the left ideal generated by $\{a_0 - \kappa^{-1} j, l - 1\} \cup \{\beta_n, \gamma_{n+1}, a_{n+1}\}_{n=0}^\infty$. $\mathcal{W}^{\kappa, j}$ is an irreducible representation of $\mathfrak{w}$. Let $w^{\kappa, j}$ be the image of 1 in $\mathcal{W}^{\kappa, j}$.

Using

$$\partial \phi(z) = \sum_{n \in \mathbb{Z}} \frac{a_n}{z^{n+1}}, \quad \beta(z) = \sum_{n \in \mathbb{Z}} \frac{\beta_n}{z^{n+1}}, \quad \gamma(z) = \sum_{n \in \mathbb{Z}} \frac{\gamma_n}{z^n}.$$ 

we define the $\hat{\mathfrak{sl}}_2$ currents with $k = \kappa^2 - 2$:

$$J^+(z) = \beta(z), \quad J^0(z) = : \gamma(z) \beta(z) : + \kappa \partial \phi(z),$$

$$J^-(z) = - : \gamma(z)^2 \beta(z) : - 2 \kappa \gamma(z) \partial \phi(z) - k \partial \gamma(z).$$

Sugawara construction with the fields above gives:

$$T(z) = - : \beta(z) \partial \gamma(z) : + : \partial \phi(z) \partial \phi(z) : - \kappa^{-1} \partial^2 \phi(z).$$
- The $\mathfrak{w}$-module $\widetilde{\mathcal{W}}^{\kappa,j}$ is defined as the quotient of $U\mathfrak{w}$ by the left ideal generated by $\{a_0 + \kappa^{-1}(j + 1), l - 1\} \cup \{\beta_{n+1}, \gamma_n, a_{n+1}\}_{n=0}^{\infty}$. Let $\widetilde{w}_{\kappa,j}$ be the image of 1 in $\widetilde{\mathcal{W}}^{\kappa,j}$. 

- Proposition: If $(k, j)$ are such that $V^{k,j}$ is irreducible, then the $\hat{\mathfrak{sl}}_2$-module maps $s: V^{k,j} \to \mathcal{W}^{\kappa,j}$, $\tilde{s}: V^{k,j} \to \widetilde{\mathcal{W}}^{\kappa,j}$ determined by $s(v^{k,j}) = w^{\kappa,j}$ and $\tilde{s}(v^{k,j}) = \tilde{w}_{\kappa,j}$, respectively, are bijections.
The \( \mathfrak{w} \)-module \( \tilde{W}^{\kappa,j} \) is defined as the quotient of \( U\mathfrak{w} \) by the left ideal generated by \( \{a_0 + \kappa^{-1}(j + 1), I - 1\} \cup \{\beta_{n+1}, \gamma_n, a_{n+1}\}_{n=0}^\infty \). Let \( \tilde{w}_{\kappa,j} \) be the image of \( 1 \) in \( \tilde{W}^{\kappa,j} \).

Fields \( J^a \) in \( \tilde{W}^{\kappa,j} \) are defined by

\[
J^+(z) = : \gamma(z)^2 \beta(z) : + 2\kappa \gamma(z) \partial \phi(z) + k \partial \gamma(z),
\]

\[
J^0(z) = - : \gamma(z) \beta(z) : - k \partial \phi(z), \quad J^-(z) = - \beta(z).
\]

This redefinition does not affect the formula for the Sugawara field.
- $\mathfrak{w}$-module $\tilde{\mathcal{W}}^{\kappa,j}$ is defined as the quotient of $U\mathfrak{w}$ by the left ideal generated by $\{a_0 + \kappa^{-1}(j + 1), l - 1\} \cup \{\beta_{n+1}, \gamma_n, a_{n+1}\}_{n=0}^\infty$. Let $\tilde{\mathcal{W}}_{\kappa,j}$ be the image of 1 in $\tilde{\mathcal{W}}^{\kappa,j}$.

- Fields $J^a$ in $\tilde{\mathcal{W}}^{\kappa,j}$ are defined by

$$J^+(z) = \gamma(z)^2 \beta(z) : +2\kappa \gamma(z) \partial \phi(z) + k \partial \gamma(z),$$

$$J^0(z) = - : \gamma(z) \beta(z) : -\kappa \partial \phi(z),$$

$$J^-(z) = -\beta(z).$$

This redefinition does not affect the formula for the Sugawara field.
\[ w \text{-module } \widetilde{W}^{\kappa,j} \text{ is defined as the quotient of } Uw \text{ by the left ideal generated by } \{a_0 + \kappa^{-1}(j + 1), l - 1\} \cup \{\beta_{n+1}, \gamma_n, a_{n+1}\}_{n=0}^\infty. \text{ Let } \widetilde{w}_{\kappa,j} \text{ be the image of } 1 \text{ in } \widetilde{W}^{\kappa,j}.

\text{Fields } J^a \text{ in } \widetilde{W}^{\kappa,j} \text{ are defined by}
\begin{align*}
J^+(z) &= : \gamma(z)^2 \beta(z) : + 2\kappa \gamma(z) \partial \phi(z) + k \partial \gamma(z), \\
J^0(z) &= - : \gamma(z) \beta(z) : - \kappa \partial \phi(z), \\
J^-(z) &= - \beta(z).
\end{align*}

This redefinition does not affect the formula for the Sugawara field.

**Proposition**

If \((k,j)\) are such that \(V^{k,j}\) is irreducible, then the \(\widehat{sl}_2\)-module maps
\[ s : V^{k,j} \to W^{\kappa,j}, \quad \tilde{s} : V^{k,j} \to \widetilde{W}^{\kappa,j}, \]
determined by \(s(v_{k,j}) = w_{\kappa,j}\) and \(\tilde{s}(v_{k,j}) = \widetilde{w}_{\kappa,j}\), respectively, are bijections.
In the space $\otimes \mathcal{V}^{k,j,\epsilon} = \mathcal{V}^{k,j} \otimes \mathcal{H}^{1,\epsilon}$ we have two commuting sets of currents: $J^a(z)$ inherited from $\mathcal{V}^{k,j}$ and $K^a(z)$ inherited from $\mathcal{H}^{1,\epsilon}$. The combined currents

$$J^a(z) = J^a(z) + K^a(z) = \sum_{n \in \mathbb{Z}} \frac{\mathcal{J}^a_n}{z^{n+1}}$$

make $\otimes \mathcal{V}^{k,j,\epsilon}$ a representation of $\widehat{\mathfrak{sl}_2}$ at level $k + 1$. 

Leszek Hadasz

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In the space $\otimes V^{k,j,\epsilon} = V^{k,j} \otimes_{\overline{C}} \mathcal{H}^{1,\epsilon}$ we have two commuting sets of currents: $J^a(z)$ inherited from $V^{k,j}$ and $K^a(z)$ inherited from $\mathcal{H}^{1,\epsilon}$. The combined currents

$$J^a(z) = J^a(z) + K^a(z) = \sum_{n \in \mathbb{Z}} \frac{J_n^a}{z^{n+1}}$$

make $\otimes V^{k,j,\epsilon}$ a representation of $\widehat{sl}_2$ at level $k + 1$.

Let $k \not\in \{-2,-3\}$ and let $T^J$, $T^K$ and $T^J$ be the Sugawara fields constructed from the respective currents. Then

$$T^{Vir}(z) = T^J(z) + T^K(z) - T^J(z) = \sum_{n \in \mathbb{Z}} \frac{L^{Vir}_n}{z^{n+2}}.$$ 

is the Virasoro current with central charge $c^\otimes_k = c_k + c_1 - c_{k+1}$. It commutes with $J^a(z)$, hence $V^{k,j,\epsilon}$ is a representation of $\widehat{sl}_2 \oplus Vir$, where $Vir$ is the Virasoro algebra spanned by $L^{Vir}_n$. 
Tensor product: free field modules

Let $\mathcal{W}^{\kappa,j,\epsilon} = \mathcal{W}^{\kappa,j} \otimes_{\mathbb{C}} \mathcal{H}^{1,\epsilon}$ and $\widetilde{\mathcal{W}}^{\kappa,j,\epsilon} = \widetilde{\mathcal{W}}^{\kappa,j} \otimes_{\mathbb{C}} \mathcal{H}^{1,\epsilon}$. 

Proposition $w_n^{\kappa,j,\epsilon}$ and $\widetilde{w}_n^{\kappa,j,\epsilon}$ are nonzero elements annihilated by $\{J_{a|n}, L_{\text{Vir}} n\}_{n > 0}$ and $J_0$ and $J_{00}$.
Tensor product: free field modules

Let $\otimes \mathcal{W}_{\kappa,j}^{\kappa,j}\otimes_{\mathbb{C}} \mathcal{H}^{1,\epsilon}$ and $\otimes \tilde{\mathcal{W}}_{\kappa,j}^{\kappa,j}\otimes_{\mathbb{C}} \mathcal{H}^{1,\epsilon}$.

Define

$$\sum_{n \in \mathbb{Z}} \frac{\rho_n}{z^{n+1}} = \rho(z) = -\gamma^2(z)K^+(z) + 2\gamma(z)K^0(z) + K^-(z) + \partial \gamma(z) \text{ in } \otimes \mathcal{W}_{\kappa,j}^{\kappa,j},$$

$$\sum_{n \in \mathbb{Z}} \frac{\tilde{\rho}_n}{z^{n+1}} = \tilde{\rho}(z) = -\gamma^2(z)K^-(z) + 2\gamma(z)K^0(z) + K^+(z) - \partial \gamma(z) \text{ in } \otimes \tilde{\mathcal{W}}_{\kappa,j}^{\kappa,j}.$$

and let for any $n \in \mathbb{N}$:

$$\mathcal{W}_{\kappa,j}^{\kappa,j} = \rho_{-2n-1} \cdots \rho_{-1}(w_{\kappa,j} \otimes f_\epsilon) \in \otimes \mathcal{W}_{\kappa,j}^{\kappa,j},$$

$$\tilde{\mathcal{W}}_{\kappa,j}^{\kappa,j} = \tilde{\rho}_{-2n-1} \cdots \tilde{\rho}_{-1}(\tilde{w}_{\kappa,j} \otimes f_\epsilon) \in \otimes \tilde{\mathcal{W}}_{\kappa,j}^{\kappa,j}.$$
Tensor product: free field modules

Let $\otimes \mathcal{W}^{\kappa,j,\epsilon} = \mathcal{W}^{\kappa,j} \otimes_{\mathbb{C}} \mathcal{H}^{1,\epsilon}$ and $\otimes \tilde{\mathcal{W}}^{\kappa,j,\epsilon} = \tilde{\mathcal{W}}^{\kappa,j} \otimes_{\mathbb{C}} \mathcal{H}^{1,\epsilon}$.

Define

$$\sum_{n \in \mathbb{Z}} \frac{\rho_n}{z^{n+1}} = \rho(z) = -\gamma^2(z)K^+(z) + 2\gamma(z)K^0(z) + K^-(z) + \partial \gamma(z) \quad \text{in} \quad \otimes \mathcal{W}^{\kappa,j,\epsilon},$$

$$\sum_{n \in \mathbb{Z}} \frac{\tilde{\rho}_n}{z^{n+1}} = \tilde{\rho}(z) = -\gamma^2(z)K^-(z) + 2\gamma(z)K^0(z) + K^+(z) - \partial \gamma(z) \quad \text{in} \quad \otimes \tilde{\mathcal{W}}^{\kappa,j,\epsilon}.$$ 

and let for any $n \in \mathbb{N}$:

$$w_n^{\kappa,j,\epsilon} = \rho_{-2n-1} \cdots \rho_{-1} w_{\kappa,j} \otimes f_{\epsilon} \in \otimes \mathcal{W}^{\kappa,j,\epsilon},$$

$$\tilde{w}_n^{\kappa,j,\epsilon} = \tilde{\rho}_{-2n-1} \cdots \tilde{\rho}_{-1} \tilde{w}_{\kappa,j} \otimes f_{\epsilon} \in \otimes \tilde{\mathcal{W}}^{\kappa,j,\epsilon}.$$

**Proposition**

\[w_n^{\kappa,j,\epsilon}, \tilde{w}_n^{\kappa,j,\epsilon}\] are nonzero elements annihilated by \(\{\mathcal{J}_n^a, L_n^{\text{Vir}}\}_{n>0}\) and \(\mathcal{J}_0^+\) and \(\mathcal{J}_0\)

\[
\begin{align*}
\mathcal{J}_0 w_n^{\kappa,j,\epsilon} &= (j + \epsilon - n) w_n^{\kappa,j,\epsilon}, \\
\mathcal{J}_0 \tilde{w}_n^{\kappa,j,\epsilon} &= (j + \epsilon + n) \tilde{w}_n^{\kappa,j,\epsilon}, \\
L_0^{\text{Vir}} w_n^{\kappa,j,\epsilon} &= (\Delta^{\otimes}_{k,j,\epsilon} + n^2) w_n^{\kappa,j,\epsilon}, \\
L_0^{\text{Vir}} \tilde{w}_n^{\kappa,j,\epsilon} &= (\Delta^{\otimes}_{k,j,\epsilon} + n^2) \tilde{w}_n^{\kappa,j,\epsilon},
\end{align*}
\]

where \(\Delta^{\otimes}_{k,j,\epsilon} = \Delta_{k,j} + \Delta_{1,\epsilon} - \Delta_{k+1,j+\epsilon}\).
Homomorphisms $s, \tilde{s}$ induce maps

$$s : \bigotimes V^{k,j,\epsilon} \to \bigotimes V^{\kappa,j,\epsilon}, \quad \tilde{s} : \bigotimes V^{k,j,\epsilon} \to \bigotimes \tilde{V}^{\kappa,j,\epsilon}.$$ 

We have

**Proposition**

For every $\kappa \neq 0$, $n \in \mathbb{N}$ and $\epsilon \in \{0, \frac{1}{2}\}$ there exist polynomials $p^n_{\kappa,\epsilon}(j), \tilde{p}^n_{\kappa,\epsilon}(j)$ in $j$, unique up to scalars, such that

$$v^n_{\kappa,j,\epsilon} = p^n_{\kappa,\epsilon}(j)s^{-1}(w^n_{\kappa,j,\epsilon}), \quad \tilde{v}^n_{\kappa,j,\epsilon} = \tilde{p}^n_{\kappa,\epsilon}(j)\tilde{s}^{-1}(\tilde{w}^n_{\kappa,j,\epsilon})$$

are defined and nonzero for every $j \in \mathbb{C}$. These are highest weight vectors satisfying eigenvalue equations for $J^0_0$ and, if $k \neq -3$ so that $T^{\text{Vir}}$ is defined, also eigenvalue equations for $L^\text{Vir}_0$. 
Homomorphisms $s, \tilde{s}$ induce maps

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One can write down the form of $p^n_{\kappa,\epsilon}(j)$ and $\tilde{p}^n_{\kappa,\epsilon}(j)$ explicitly. The calculation requires the knowledge of determinants of maps $s$ and $\tilde{s}$, some information about the Wakimoto representation of the $\hat{sl}_2$ singular vectors and uses special properties of vectors $w^n_{\kappa,j,\epsilon}, \tilde{w}^n_{\kappa,j,\epsilon}$. 
The relevant CFT-s

The CFT models with chiral symmetry given by $\hat{\mathfrak{sl}}_2$ are well known:

- the $\hat{\mathfrak{su}}(2)_k$ WZNW model, equivalent, under the general $G \leftrightarrow G^C/G$ duality, to the $H^+_3 = SL(2, \mathbb{C})/SU(2)$ coset model at the level $k' = -k$; the structure constants of the latter have been calculated by the conformal bootstrap method [Teschner];
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- its spectrum corresponds to non-unitary, infinite dimensional $\mathfrak{su}(2)$ representations with $j \in -\frac{1}{2} + i\mathbb{R}$;
- even if the $\hat{\mathfrak{su}}(2)_{k<2}$ structure constants do not admit an analytic continuation to the region $k > -2$, one can analytically continue the difference bootstrap equations and derive the structure constants of the “imaginary” $\hat{\mathfrak{su}}(2)$ WZNW model [Dabholkar, Pakman];
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- even if the $\hat{su}(2)_k<_{-2}$ structure constants do not admit an analytic continuation to the region $k > -2$, one can analytically continue the difference bootstrap equations and derive the structure constants of the “imaginary” $\hat{su}(2)$ WZNW model [Dabholkar,Pakman];
- we also have full information about the Liouville field theory, with the symmetry algebra given by $Vir$. 

Leszek Hadasz

Non-rational $\hat{su}(2)$ cosets and Liouville field theory
The CFT state–operator map

\[ \otimes V^{k,j,\epsilon} \ni \nu_{\kappa,j,\epsilon} \mapsto \Phi(\nu_{\kappa,j,\epsilon}|z) \in \text{Hom}(\otimes V^{k,1,\epsilon}, \otimes V^{k,3,\epsilon}) \]

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\begin{align*}
\otimes \mathcal{W}^{k,j,\epsilon} & \ni w_{k,j,\epsilon}^{n} \mapsto \Phi (w_{k,j,\epsilon}^{n} | z) \in \text{Hom}(\otimes \mathcal{W}^{k,j_1,\epsilon}, \otimes \mathcal{W}^{k,j_3,\epsilon}) \\
\uparrow
\otimes \mathcal{V}^{k,j,\epsilon} & \ni v_{k,j,\epsilon}^{n} \mapsto \Phi (v_{k,j,\epsilon}^{n} | z) \in \text{Hom}(\otimes \mathcal{V}^{k,j_1,\epsilon}, \otimes \mathcal{V}^{k,j_3,\epsilon}) \\
\downarrow
\otimes \tilde{\mathcal{W}}^{k,j,\epsilon} & \ni \tilde{w}_{k,j,\epsilon}^{n} \mapsto \Phi (\tilde{w}_{k,j,\epsilon}^{n} | z) \in \text{Hom}(\otimes \tilde{\mathcal{W}}^{k,j_1,\epsilon}, \otimes \tilde{\mathcal{W}}^{k,j_3,\epsilon})
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The CFT state–operator map

\[ \otimes \mathcal{W}^{\kappa, j, \epsilon} \ni w_{\kappa, j, \epsilon} \mapsto \Phi \left( w_{\kappa, j, \epsilon} \bigg| z \right) \in \text{Hom}\left( \otimes \mathcal{W}^{k_1, \epsilon}, \otimes \mathcal{W}^{k_3, \epsilon} \right) \]

\[ \uparrow \]

\[ \otimes \mathcal{V}^{\kappa, j, \epsilon} \ni v_{\kappa, j, \epsilon} \mapsto \Phi \left( v_{\kappa, j, \epsilon} \bigg| z \right) \in \text{Hom}\left( \otimes \mathcal{V}^{k_1, \epsilon}, \otimes \mathcal{V}^{k_3, \epsilon} \right) \]

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\[ \otimes \widetilde{\mathcal{W}}^{\kappa, j, \epsilon} \ni \widetilde{w}_{\kappa, j, \epsilon} \mapsto \Phi \left( \widetilde{w}_{\kappa, j, \epsilon} \bigg| z \right) \in \text{Hom}\left( \otimes \widetilde{\mathcal{W}}^{k_1, \epsilon}, \otimes \widetilde{\mathcal{W}}^{k_3, \epsilon} \right) \]

- The \( \widehat{\mathfrak{su}}(2)_k \times \widehat{\mathfrak{su}}(2)_1 \) Ward identities allow to express correlation function of three \( \Phi \left( v_{\kappa, j, \epsilon} \big| z \right) \) fields as a product of the three-point conformal block \( \rho[\ldots] \) and structure constants of the models \( \widehat{\mathfrak{su}}(2)_k \times \widehat{\mathfrak{su}}(2)_1 \).
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\[ \otimes \mathcal{W}^{\kappa,j,\epsilon} \ni w^{n}_{\kappa,j,\epsilon} \mapsto \Phi (w^{n}_{\kappa,j,\epsilon} | z) \in \text{Hom}(\otimes \mathcal{W}^{k_{1},\epsilon}, \otimes \mathcal{W}^{k_{3},\epsilon}) \]

\[ \uparrow \]

\[ \otimes \mathcal{Y}^{k,j,\epsilon} \ni v^{n}_{\kappa,j,\epsilon} \mapsto \Phi (v^{n}_{\kappa,j,\epsilon} | z) \in \text{Hom}(\otimes \mathcal{Y}^{k_{1},\epsilon}, \otimes \mathcal{Y}^{k_{3},\epsilon}) \]

\[ \downarrow \]

\[ \otimes \tilde{\mathcal{W}}^{k,j,\epsilon} \ni \tilde{w}^{n}_{\kappa,j,\epsilon} \mapsto \Phi (\tilde{w}^{n}_{\kappa,j,\epsilon} | z) \in \text{Hom}(\otimes \tilde{\mathcal{W}}^{k_{1},\epsilon}, \otimes \tilde{\mathcal{W}}^{k_{3},\epsilon}) \]

- The \( \hat{su}(2)_{k} \times \hat{su}(2)_{1} \) Ward identities allow to express correlation function of three \( \Phi (v^{n}_{\kappa,j,\epsilon} | z) \) fields as a product of the three-point conformal block \( \rho[\ldots] \) and structure constants of the models \( \hat{su}(2)_{k} \times \hat{su}(2)_{1} \);

- one can compute explicitly the block \( \rho[\ldots] \) and check, that the correlation function of three \( \Phi (v^{n}_{\kappa,j,\epsilon} | z) \) fields is equal to the product of structure consultants of the \( \hat{su}(2)_{k+1} \times \text{Vir} \) model.
The equivalence

\[ \hat{\mathfrak{su}}(2)_k \times \hat{\mathfrak{su}}(2)_1 \sim \hat{\mathfrak{su}}(2)_{k+1} \times \text{Vir} \]

is suggested by the GKO coset construction of the minimal models

\[ V(1, m) \sim \frac{\hat{\mathfrak{su}}(2)_m \times \hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_{m+1}}, \]
The equivalence
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is suggested by the GKO coset construction of the minimal models
\[ V(1, m) \sim \frac{\hat{su}(2)_m \times \hat{su}(2)_1}{\hat{su}(2)_{m+1}} , \]
Since
\[ SV(m) \times V(1) \sim \frac{\hat{su}(2)_m \times \hat{su}(2)_2}{\hat{su}(2)_{m+2}} \times \frac{\hat{su}(2)_1 \times \hat{su}(2)_1}{\hat{su}(2)_2} \]
\[ V(m+1) \times V(m) \sim \frac{\hat{su}(2)_{m+1} \times \hat{su}(2)_1}{\hat{su}(2)_{m+2}} \times \frac{\hat{su}(2)_m \times \hat{su}(2)_1}{\hat{su}(2)_{m+1}} \]
then, if we relax the condition that \( m \) is an integer, then the formula
\[ SVir \times V(1) \sim \text{Vir} \times \text{Vir}' \]
relating an \( N = 1 \) superconformal Liouville field theory (times a free fermion) to a pair of Liouville fields, is also expected (and holds).
There are different possible extensions of this results:

- One may expect that the continuous spectra generalization of the GKO construction will work for the $N=1$ supersymmetric case

$$\hat{su}(2)_k \times \hat{su}(2)_2 \sim N=1 \text{ super-Liouville} \times \hat{su}(2)_{k+2}.$$ 

and, more generally, for the general $\hat{su}(2)$ quotients involving the para-Liouville theories:

$$\hat{su}(2)_k \times \hat{su}(2)_p \sim \text{para-Liouville} \times \hat{su}(2)_{k+p}, \quad p > 2.$$
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- The findings may be interesting for the TFT$_3$/CFT$_2$ and AdS$_3$/CFT$_2$ relations.