# Non-rational $\widehat{su}(2)$ cosets and Liouville field theory

Leszek Hadasz

Institute of Theoretical Physics
Jagiellonian University, Kraków, Poland

Results obtained in collaboration with Błażej Ruba

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## Lemma

A tensor product of the highest weight representations of the affine $\widehat{\mathfrak{sl}}_2$ algebra: the Verma module $\mathcal{V}^{k,j}$ and $\mathcal{H}^{1,\epsilon} = \mathcal{V}^{1,\epsilon}/\mathrm{rad}\,\mathcal{V}^{1,\epsilon}$ with $\epsilon = 0, \frac{1}{2}$, decomposes as:

$$\mathcal{V}^{k,j} \otimes \mathcal{H}^{1,\epsilon} \cong \bigoplus_{n\in\mathbb{Z}} \mathcal{V}^{k+1,j_n} \otimes \mathsf{V}^{c,\Delta_n},$$

where $\mathsf{V}^{c,\Delta_n}$ is the Virasoro Verma module.

- $\widehat{\mathfrak{sl}_2}$ is a Lie algebra with basis $\{J_n^{\pm}, J_n^0\}_{n\in\mathbb{Z}} \cup \{K\}$ and nonzero commutators

$$[J_n^0, J_m^{\pm}] = \pm J_{n+m}^{\pm}, \quad [J_n^+, J_m^-] = 2J_{n+m}^0 + n\delta_{n+m}K \quad [J_n^0, J_m^0] = \frac{n}{2}\delta_{n+m}K.$$

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- Let $\widehat{\mathfrak{sl}_2^+}$ be the subalgebra of $\widehat{\mathfrak{sl}_2}$ generated by $J_0^+$ and $\{J_n^a\}_{n\geq 1}$.
Verma module $\mathcal{V}^{k,j}$ is the quotient of the universal enveloping algebra of $\widehat{\mathfrak{sl}_2}$ by the left ideal generated by $\widehat{\mathfrak{sl}_2^+} \cup \{K-k, J_0^0 - j\}$. The image of $\mathbf{1}$ in $\mathcal{V}^{k,j}$ is denoted by $v_{k,j}$.

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- Every irreducible, highest weight module over $\widehat{\mathfrak{sl}_2}$ is isomorphic to some (unique) $\mathcal{H}^{k,j} = \mathcal{V}^{k,j}/\mathrm{rad}(\mathcal{V}^{k,j})$, where $\mathrm{rad}(\mathcal{V}^{k,j})$ is the union of all proper submodules of $\mathcal{V}^{k,j}$.

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- Every irreducible, highest weight module over $\widehat{\mathfrak{sl}_2}$ is isomorphic to some (unique) $\mathcal{H}^{k,j} = \mathcal{V}^{k,j}/\mathrm{rad}(\mathcal{V}^{k,j})$, where $\mathrm{rad}(\mathcal{V}^{k,j})$ is the union of all proper submodules of $\mathcal{V}^{k,j}$.
- This explains $\mathcal{V}^{k,j} \otimes \mathcal{H}^{1,\epsilon}$.

- Withe the help of the local currents $J^a(z) = \sum_{n\in\mathbb{Z}} \frac{J^a_n}{z^{n+1}}$. one defines for $k \neq -2$ the Sugawara field:

$$T(z) = \frac{1}{k+2}\left( : J^0(z)J^0(z) : + \frac{1}{2} : J^+(z)J^-(z) : + \frac{1}{2} : J^-(z)J^+(z) \right).$$

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- Its modes, defined as $T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$ generate the Virasoro algebra Vir with the central charge $c_k = \frac{3k}{k+2}$.

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- Let $\mathrm{Vir}^+$ be the subalgebra of Vir generated by $\{L_n\}_{n\geq 1}$.
  Verma module $V^{c,\Delta}$is the quotient of the universal enveloping algebra of Vir by the left ideal generated by $\mathrm{Vir}^+ \cup \{L_0 - \Delta\}$.

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- This (partially) explains $\mathcal{V}^{k+1,j_n} \otimes \mathsf{V}^{c,\Delta_n}$ in

$$\mathcal{V}^{k,j} \otimes \mathcal{H}^{1,\epsilon} \cong \bigoplus_{n\in\mathbb{Z}} \mathcal{V}^{k+1,j_n} \otimes \mathsf{V}^{c,\Delta_n},$$

- Let $\mathfrak{f}$ be the Lie superalgebra spanned by odd $\{\psi^i_n, \overline{\psi}^i_n\}^{i=1,2}_{n\in\mathbb{Z}+\frac{1}{2}}$ and even $I$, with only nonzero superbrackets

$$[\psi^i_n, \overline{\psi}^j_m] = \delta_{n+m,0}\delta^{i,j} I.$$

The Fock space $\mathcal{F}$ is defined as the quotient of $\mathrm{U}\mathfrak{f}$ by the left ideal generated by $\{\psi^i_n, \overline{\psi}^i_n\}^{i=1,2}_{n>0} \cup \{I - 1\}$. $f_0$ is the image of $1$ in $\mathcal{F}$ and $f_{\frac{1}{2}} = \overline{\psi}^1_{-\frac{1}{2}} f_0$.

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- $\mathcal{F}$ is acted upon by local fields $\psi^i(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\psi^i_n}{z^{n+\frac{1}{2}}}$, $\overline{\psi}^i(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\overline{\psi}^i_n}{z^{n+\frac{1}{2}}}$.
Modes of the currents:
$$K^a(z) = \sum_{i,j=1}^{2} \tau^a_{i,j} :\overline{\psi}^i(z)\psi^j(z):,$$
where $\tau^0 = \frac{1}{2}\sigma^3$, $\tau^{\pm} = \frac{1}{2}\left(\sigma^1 \pm i\sigma^2\right)$, satisfy the $\widehat{\mathfrak{sl}}_2$ algebra with $k=1$.

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**Proposition**

$\widehat{\mathfrak{sl}}_2$-submodule of $\mathcal{F}$ generated by $f_\epsilon$ is isomorphic to $\mathcal{H}^{1,\epsilon}$.

- Fix $\kappa \neq 0$ and let $\mathfrak{w}$ be the Lie algebra spanned by $a_n, \beta_n, \gamma_n,\ n \in \mathbb{Z}$, with commutators

$$[\gamma_m, \beta_n] = \delta_{m+n} I, \qquad [a_m, a_n] = \frac{m}{2}\delta_{m+n} I.$$

# Free fields: Wakimoto representation

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- We define a $\mathfrak{w}$-module $\mathcal{W}^{\kappa,j}$ as the quotient of $\mathrm{U}\mathfrak{w}$ by the left ideal generated by $\{a_0 - \kappa^{-1} j, I - 1\} \cup \{\beta_n, \gamma_{n+1}, a_{n+1}\}_{n=0}^{\infty}$. $\mathcal{W}^{\kappa,j}$ is an irreducible representation of $\mathfrak{w}$. Let $w_{\kappa,j}$ be the image of $1$ in $\mathcal{W}^{\kappa,j}$.

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- Using

$$\partial\phi(z) = \sum_{n\in\mathbb{Z}} \frac{a_n}{z^{n+1}}, \qquad \beta(z) = \sum_{n\in\mathbb{Z}} \frac{\beta_n}{z^{n+1}}, \qquad \gamma(z) = \sum_{n\in\mathbb{Z}} \frac{\gamma_n}{z^n}.$$

we define the $\widehat{\mathfrak{sl}}_2$ currents with $k = \kappa^2 - 2$ :

$$J^+(z) = \beta(z), \qquad J^0(z) =: \gamma(z)\beta(z) : + \kappa\partial\phi(z),$$
$$J^-(z) = - : \gamma(z)^2 \beta(z) : -2\kappa\gamma(z)\partial\phi(z) - k\partial\gamma(z).$$

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**Sugawara construction with the fields above gives:**

$$T(z) = - : \beta(z)\partial\gamma(z) : + : \partial\phi(z)\partial\phi(z) : -\kappa^{-1}\partial^2\phi(z).$$

# Free fields: dual Wakimoto module

- $\mathfrak{w}$-module $\widetilde{\mathcal{W}}^{\kappa,j}$ is defined as the quotient of $\mathrm{U}\mathfrak{w}$ by the left ideal generated by $\{a_0 + \kappa^{-1}(j+1), I - 1\} \cup \{\beta_{n+1}, \gamma_n, a_{n+1}\}_{n=0}^{\infty}$. Let $\widetilde{w}_{\kappa,j}$ be the image of $1$ in $\widetilde{\mathcal{W}}^{\kappa,j}$.

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- Fields $J^a$ in $\widetilde{\mathcal{W}}^{\kappa,j}$ are defined by

$$J^+(z) =: \gamma(z)^2\beta(z) : +2\kappa\gamma(z)\partial\phi(z) + k\partial\gamma(z),$$
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This redefinition does not affect the formula for the Sugawara field.

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**Proposition**

If $(k,j)$ are such that $\mathcal{V}^{k,j}$ is irreducible, then the $\widehat{\mathfrak{sl}}_2$-module maps

$$s : \mathcal{V}^{k,j} \to \mathcal{W}^{\kappa,j}, \qquad \widetilde{s} : \mathcal{V}^{k,j} \to \widetilde{\mathcal{W}}^{\kappa,j},$$

determined by $s(v_{k,j}) = w_{\kappa,j}$ and $\widetilde{s}(v_{k,j}) = \widetilde{w}_{\kappa,j}$, respectively, are bijections.

# Tensor product

- In the space ${}^{\otimes}\mathcal{V}^{k,j,\epsilon} = \mathcal{V}^{k,j} \otimes_{\mathbb{C}} \mathcal{H}^{1,\epsilon}$ we have two commuting sets of currents: $J^a(z)$ inherited from $\mathcal{V}^{k,j}$ and $K^a(z)$ inherited from $\mathcal{H}^{1,\epsilon}$. The combined currents

$$\mathcal{J}^a(z) = J^a(z) + K^a(z) = \sum_{n\in\mathbb{Z}} \frac{\mathcal{J}^a_n}{z^{n+1}}$$

make ${}^{\otimes}\mathcal{V}^{k,j,\epsilon}$ a representation of $\widehat{\mathfrak{sl}_2}$ at level $k+1$.

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make ${}^{\otimes}\mathcal{V}^{k,j,\epsilon}$ a representation of $\widehat{\mathfrak{sl}}_2$ at level $k+1$.
- Let $k \notin \{-2,-3\}$ and let $T^J$, $T^K$ and $T^{\mathcal{J}}$ be the Sugawara fields constructed from the respective currents. Then
$$T^{\mathrm{Vir}}(z) = T^J(z) + T^K(z) - T^{\mathcal{J}}(z) = \sum_{n\in\mathbb{Z}} \frac{L^{\mathrm{Vir}}_n}{z^{n+2}}.$$
is the Virasoro current with central charge $c^{\otimes}_k = c_k + c_1 - c_{k+1}$. It commutes with $\mathcal{J}^a(z)$, hence $\mathcal{V}^{k,j,\epsilon}$ is a representation of $\widehat{\mathfrak{sl}}_2 \oplus \mathrm{Vir}$, where $\mathrm{Vir}$ is the Virasoro algebra spanned by $L^{\mathrm{Vir}}_n$.

## Tensor product: free field modules

Let ${}^{\otimes}\mathcal{W}^{\kappa,j,\epsilon} = \mathcal{W}^{\kappa,j} \otimes_{\mathbb{C}} \mathcal{H}^{1,\epsilon}$ and ${}^{\otimes}\widetilde{\mathcal{W}}^{\kappa,j,\epsilon} = \widetilde{\mathcal{W}}^{\kappa,j} \otimes_{\mathbb{C}} \mathcal{H}^{1,\epsilon}$.

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Define

$$\sum_{n\in\mathbb{Z}} \frac{\rho_n}{z^{n+1}} = \rho(z) = -\gamma^2(z)K^+(z) + 2\gamma(z)K^0(z) + K^-(z) + \partial\gamma(z) \quad \text{in } {}^{\otimes}\mathcal{W}^{\kappa,j,\epsilon},$$

$$\sum_{n\in\mathbb{Z}} \frac{\widetilde{\rho}_n}{z^{n+1}} = \widetilde{\rho}(z) = -\gamma^2(z)K^-(z) + 2\gamma(z)K^0(z) + K^+(z) - \partial\gamma(z) \quad \text{in } {}^{\otimes}\widetilde{\mathcal{W}}^{\kappa,j,\epsilon}.$$

and let for any $n \in \mathbb{N}$ :

$$w^n_{\kappa,j,\epsilon} = \rho_{-2n-1}\cdots\rho_{-1}(w_{\kappa,j} \otimes f_\epsilon) \in {}^{\otimes}\mathcal{W}^{\kappa,j,\epsilon},$$

$$\widetilde{w}^n_{\kappa,j,\epsilon} = \widetilde{\rho}_{-2n-1}\cdots\widetilde{\rho}_{-1}(\widetilde{w}_{\kappa,j} \otimes f_\epsilon) \in {}^{\otimes}\widetilde{\mathcal{W}}^{\kappa,j,\epsilon}.$$

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$$w^n_{\kappa,j,\epsilon} = \rho_{-2n-1}\cdots\rho_{-1}(w_{\kappa,j} \otimes f_\epsilon) \in {}^{\otimes}\mathcal{W}^{\kappa,j,\epsilon},$$
$$\widetilde{w}^n_{\kappa,j,\epsilon} = \widetilde{\rho}_{-2n-1}\cdots\widetilde{\rho}_{-1}(\widetilde{w}_{\kappa,j} \otimes f_\epsilon) \in {}^{\otimes}\widetilde{\mathcal{W}}^{\kappa,j,\epsilon}.$$

## Proposition

$w^n_{\kappa,j,\epsilon}, \widetilde{w}_{\kappa,j,\epsilon}$ are nonzero elements annihilated by $\{\mathcal{J}^a_n, L^{\mathrm{Vir}}_n\}_{n>0}$ and $\mathcal{J}^+_0$ and

$$\mathcal{J}^0_0 w^n_{\kappa,j,\epsilon} = (j+\epsilon-n)w^n_{\kappa,j,\epsilon}, \qquad L^{\mathrm{Vir}}_0 w^n_{\kappa,j,\epsilon} = (\Delta^{\otimes}_{k,j,\epsilon} + n^2)w^n_{\kappa,j,\epsilon}, \tag{1}$$
$$\mathcal{J}^0_0 \widetilde{w}^n_{\kappa,j,\epsilon} = (j+\epsilon+n)\widetilde{w}^n_{\kappa,j,\epsilon}, \qquad L^{\mathrm{Vir}}_0 \widetilde{w}^n_{\kappa,j,\epsilon} = (\Delta^{\otimes}_{k,j,\epsilon} + n^2)\widetilde{w}^n_{\kappa,j,\epsilon},$$

where $\Delta^{\otimes}_{k,j,\epsilon} = \Delta_{k,j} + \Delta_{1,\epsilon} - \Delta_{k+1,j+\epsilon}$.

Homomorphisms $s, \tilde{s}$ induce maps

$$\mathbf{s} : {}^{\otimes}\mathcal{V}^{k,j,\epsilon} \to {}^{\otimes}\mathcal{W}^{\kappa,j,\epsilon}, \qquad \tilde{\mathbf{s}} : {}^{\otimes}\mathcal{V}^{k,j,\epsilon} \to {}^{\otimes}\widetilde{\mathcal{W}}^{\kappa,j,\epsilon}.$$

We have

**Proposition**

For every $\kappa \neq 0$, $n \in \mathbb{N}$ and $\epsilon \in \{0, \frac{1}{2}\}$ there exist polynomials $p^n_{\kappa,\epsilon}(j), \tilde{p}^n_{\kappa,\epsilon}(j)$ in $j$, unique up to scalars, such that

$$v^n_{\kappa,j,\epsilon} = p^n_{\kappa,\epsilon}(j)\mathbf{s}^{-1}(w^n_{\kappa,j,\epsilon}), \qquad \tilde{v}^n_{\kappa,j,\epsilon} = \tilde{p}^n_{\kappa,\epsilon}(j)\tilde{\mathbf{s}}^{-1}(\tilde{w}^n_{\kappa,j,\epsilon})$$

are defined and nonzero for every $j \in \mathbb{C}$. These are highest weight vectors satisfying eigenvalue equations for $\mathcal{J}^0_0$ and, if $k \neq -3$ so that $T^{\mathrm{Vir}}$ is defined, also eigenvalue equations for $L^{\mathrm{Vir}}_0$.

# Normalization

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One can write down the form of $p^n_{\kappa,\epsilon}(j)$ and $\tilde{p}^n_{\kappa,\epsilon}(j)$ explicitly. The calculation requires the knowledge of determinants of maps $\mathbf{s}$ and $\tilde{\mathbf{s}}$, some information about the Wakimoto representation of the $\widehat{\mathfrak{sl}}_2$ singular vectors and uses special properties of vectors $w^n_{\kappa,j,\epsilon}, \tilde{w}^n_{\kappa,j,\epsilon}$.

# The relevant CFT-s

The CFT models with chiral symmetry given by $\widehat{\mathfrak{sl}}_2$ are well known:

- the $\widehat{su}(2)_k$ WZNW model, equivalent, under the general $G \leftrightarrow G^{\mathbb{C}}/G$ duality, to the $H_3^+ = SL(2,\mathbb{C})/SU(2)$ coset model at the level $k' = -k$; the structure constants of the latter have been calculated by the conformal bootstrap method [Teschner];

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- even if the $\widehat{su}(2)_{k<-2}$ structure constants do not admit an analytic continuation to the region $k > -2$, one can analytically continue the difference bootstrap equations and derive the structure constants of the "imaginary" $\widehat{su}(2)$ WZNW model [Dabholkar,Pakman];

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- we also have full information about the Liouville field theory, with the symmetry algebra given by Vir.

# The CFT state–operator map

$$^{\otimes}\mathcal{V}^{k,j,\epsilon} \ni v^{n}_{\kappa,j,\epsilon} \quad \mapsto \quad \Phi\left(v^{n}_{\kappa,j,\epsilon}\middle|z\right) \in \mathrm{Hom}\left({}^{\otimes}\mathcal{V}^{k,j_1,\epsilon}, {}^{\otimes}\mathcal{V}^{k,j_3,\epsilon}\right)$$

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- The $\widehat{su}(2)_k \times \widehat{su}(2)_1$ Ward identities allow to express correlation function of three $\Phi\left(v^n_{\kappa,j,\epsilon}\middle|z\right)$ fields as a product of the three-point conformal blok $\rho[\ldots]$ and structure constants of the models $\widehat{su}(2)_k \times \widehat{su}(2)_1$;

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- one can compute explicitly the block $\rho[\ldots]$ and check, that the correlation function of three $\Phi\left(v^n_{\kappa,j,\epsilon}\middle|z\right)$ fields is equal to the product of structure consultants of the $\widehat{su}(2)_{k+1} \times \mathrm{Vir}$ model.

The equivalence

$$\widehat{su}(2)_k \;\times\; \widehat{su}(2)_1 \;\sim\; \widehat{su}(2)_{k+1} \times \mathrm{Vir}$$

is suggested by the GKO coset construction of the minimal models

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Since

$$SV(m) \times V(1) \sim \frac{\widehat{su}(2)_m \times \widehat{su}(2)_2}{\widehat{su}(2)_{m+2}} \times \frac{\widehat{su}(2)_1 \times \widehat{su}(2)_1}{\widehat{su}(2)_2}$$

$$V(m+1) \times V(m) \sim \frac{\widehat{su}(2)_{m+1} \times \widehat{su}(2)_1}{\widehat{su}(2)_{m+2}} \times \frac{\widehat{su}(2)_m \times \widehat{su}(2)_1}{\widehat{su}(2)_{m+1}}$$

then, if we relax the condition that $m$ is an integer, then the formula

$$\mathrm{SVir} \times V(1) \sim \mathrm{Vir} \times \mathrm{Vir}'$$

relating an $\mathrm{N} = 1$ superconformal Liouville field theory (times a free fermion) to a pair of Liouville fields, is also expected (and holds).

There are different possible extensions of this results:

- One may expect that the continuous spectra generalization of the GKO construction will work for the N=1 supersymmetric case

$$\widehat{su}(2)_k \;\times\; \widehat{su}(2)_2 \;\sim\; \text{N=1 super-Liouville} \;\times\; \widehat{su}(2)_{k+2}\;.$$

  and, more generally, for the general $\widehat{su}(2)$ quotients involving the para-Liouville theories:

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- The findings may be interesting for the $TFT_3/CFT_2$ and $AdS_3/CFT_2$ relations.