The Langlands Program for 3-manifolds

Sam Gunningham, Montana State University

joint with:

David Ben-Zvi, David Jordan, Pavel Safronov

e.g. $G = SL_N$, $G^\vee = PGL_N$

Notation:

G, $G^\vee$ Langlands dual reductive groups

$\Psi = -1/\Psi^\vee \in \mathbb{C} \cup \{\infty\}$

M closed oriented 3-manifold

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The Goals:

I) To define spaces of states, $Z_{G,\Psi}(M)$ in Kapustin-Witten twist of 4d $\mathcal{N}=4$ SYM.

II)

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The Goals:

I) To define spaces of states, $Z_{G,\Psi}(M)$ in Kapustin–Witten twist of 4d $\mathcal{N}=4$ SYM.

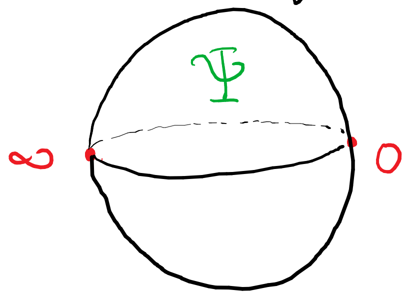
II) Verify S-duality: $Z_{G,\Psi}(M) \simeq Z_{G^\vee,\Psi^\vee}(M)$

The Ψ Landscape

B-model, $\psi = \infty$

$\mathbb{C} \cup \{\infty\}$

A-model $\psi = 0$



The Ψ Landscape

$\mathbb{C} \cup \{\infty\}$

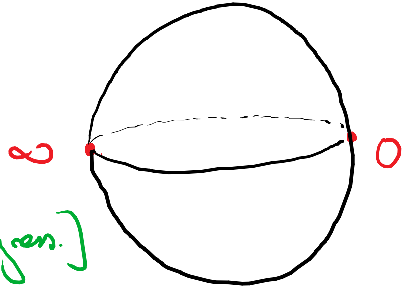
B-model, $\Psi = \infty$

aka Galois

aka Spectral

Algebraic geometry

[Beraldo-G-Safronov, in progress.]



A-model $\Psi = 0$

aka Automorphic

(symplectic) topology

[Witten, ϕ]

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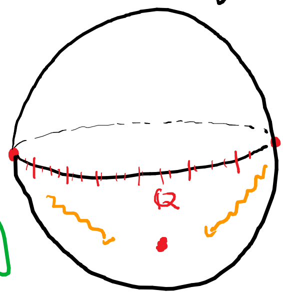

0

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[Witten, $\emptyset$]

Deformation quantization

skein theory

[GJS, In progress]

generic Ψ

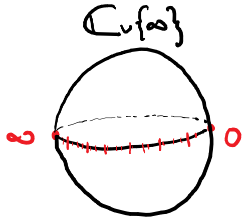
Floer-Novikov theory

(sheaf-theoretic)

[GS, in progress]

[Brav, Bussi, Dupont, Joyce, Szendrői, Abouzaid, Manolescu]

Towards a definition:



$$\mathcal{O}(\mathrm{Loc}_G(M))$$

algebraic functions on character variety

$$\mathrm{Hom}(\pi_1 M, G)/G$$

(Well-defined but needs refining)

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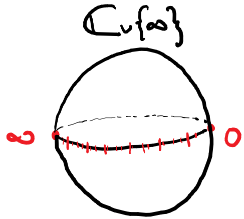

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First approx:

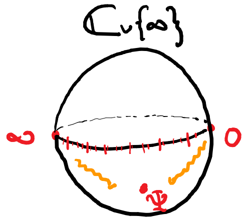
$$H_*(\mathrm{Conn}_G(M))$$

"$HF_*(\mathrm{Conn}_G(M), CS_G)$"

Morse homology of Chern-Simons functional on $G(\mathbb{C})$-connections mod gauge.

(No math. definition!)

Towards a definition:



$$\mathcal{O}(\mathrm{Loc}_G(M)) \qquad\qquad\qquad \text{“}HF_*(\mathrm{Conn}_G(M), CS_G)\text{”}$$

$$\Big\downarrow q \sim e^{-1/\Psi} \qquad\qquad\qquad\qquad \Big\downarrow ??$$

$$\mathcal{O}^q(\mathrm{Loc}_G(M)) \longleftrightarrow H^*(\mathrm{Loc}_G(M), P)$$

G-skein module: $\mathcal{O}^q(\mathrm{Loc}_G(M)) = Sk_{G,q}(M)$

Sheaf theoretic Floer homology: $H^*(\mathrm{Loc}_G(M), P) = HP^*_G(M)$

[Przytycki, Turaev, ...]

[AM, BBDJS]

More generally: expect a pair of

$(0,1,2,3)$-extended 4d-TFTs

$$\left.\begin{array}{l} Z^{(A)}_{G,\Psi} \\ \\ Z^{(B)}_{G,\Psi} \end{array}\right\} : \mathrm{Bord}_{0,1,2,3} \longrightarrow 3\text{-}\mathrm{Cat}_{\mathbb{C}}$$

s.t.

$$Z^{(A)}_{G,\Psi} \simeq Z^{(B)}_{G^\vee,\Psi^\vee} \quad (\text{S-duality})$$

Moreover, if $\underline{\Psi}$ = generic, expect:

1) $\dim H^i \mathcal{Z}_{G,\underline{\Psi}}(M) < \infty$

(independent of $\underline{\Psi}$)

c.f. Witten's finiteness conjecture:

$\boxed{\begin{array}{l}\underline{\text{Thm}} \text{ (GJS '19)} \\ \dim Sk_G^{gen}(M) < \infty\end{array}}$

2) $\mathcal{Z}^{(A)}_{G,\underline{\Psi}}(M) \cong \mathcal{Z}^{(B)}_{G,\underline{\Psi}}(M)$

A Riemann-Hilbert theorem

X (-1)-shifted symplectic

Locally $X \cong d\mathrm{Crit}(f)$, $f : U \longrightarrow \mathbb{C}$ hol.

e.g. $X = \mathrm{Loc}_G(M)$ "=" $\mathrm{Hom}(\pi_1 M, G)/G$

$\rightsquigarrow$ two perverse sheaves on X

$$P_X \qquad\qquad \left(\mathcal{O}_X((\hbar)), \ \Delta_\hbar \right)$$

Vanishing cycle sheaf, locally $P_X \cong \phi_f$

[Brav-Bussi-Dupont-Joyce-Szendroi]

(-1)-shifted (aka BV) deformation quantization

[Pridham]

__Theorem__ [G–Safronov, in progress]

$$\underset{\substack{T \\ \text{monodromy}}}{G} \circlearrowright P_X \overset{\widehat{RH}}{\rightsquigarrow} \left(\mathcal{O}_X((\hbar)), \hbar\Delta\right) \circlearrowleft \underset{\hbar\text{-connection}}{D_\hbar}$$

formal Riemann–Hilbert correspondence

Local story (Sabbah, Saito, ...):

$$\phi_f \overset{\widehat{RH}}{\rightsquigarrow} \left(\Omega_U((\hbar)), \hbar d + df\right)$$

Corollary [Not immediate!] $X = \mathrm{Loc}_G(M)$

$$\dim HP^0_G(M) = \dim Sk^{gen}_G(M)$$

$\rightsquigarrow$ Numerical conjectures:

- $\dim Sk_G(M) = \dim Sk_{G^\vee}(M)$
- $\operatorname{gr} \dim HP^*_G(M) = \operatorname{gr} \dim HP^*_{G^\vee}(M)$

[c.f. D. Jordan's talk]

Further directions:

Thanks for your attention!

- Refined B-model at $\Psi = \infty$:

$$R\Gamma\left(\mathrm{Loc}_G(M);\, \omega_{\mathcal{N}}\right)$$

[Beraldo, G, Safronov in progress]

- Atiyah-Bott duality:

$$H_*\left(\mathrm{Conn}_G(M)\right) \cong R\Gamma\left(\mathrm{Loc}_{G^\vee}(M);\, \omega_{\mathcal{N}^{reg}}\right)$$

c.f. NAPD

- Refined A-model at $\Psi = 0$:

??

[Witten: Floer-theoretic proposal]

Categories of branes:

$\boxed{\Sigma \quad \text{(Riemann) surface}}$

$$Z^{(A)}_{G,0}(\Sigma) = \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G(\Sigma))$$

topological sheaves

$$Z^{(B)}_{G,\infty}(\Sigma) = \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Loc}_G(\Sigma))$$

coherent sheaves

[Beilinson–Drinfeld, Arinkin–Gaitsgory, Ben-Zvi–Nadler, ...]

Categories of branes:

$\boxed{\Sigma \quad (\text{Riemann}) \text{ surface}}$

$$Z^{(A)}_{G,\Psi}(\Sigma) = \mathrm{Shv}^{q^\vee}_{\mathcal{N}}(\mathrm{Bun}_G(\Sigma)) \qquad q^\vee \sim e^{\Psi}$$

[Ben-Zvi–Nadler]

$$Z^{(B)}_{G,\Psi}(\Sigma) = \mathrm{IndCoh}^{q}_{\mathcal{N}}(\mathrm{Loc}_G(\Sigma)) \qquad q \sim e^{-1/\Psi}$$

[Ben-Zvi–Brochier–Jordan]

Categories of branes:

$\boxed{\Sigma \quad \text{(Riemann) surface}}$

$$Z^{(A)}_{G,\Psi}(\Sigma) = \mathrm{Shv}^{q^\vee}_{\mathcal{N}}(\mathrm{Bun}_G(\Sigma))$$

Note: uses cx structure!

$$Z^{(B)}_{G,\Psi}(\Sigma) = \mathrm{IndCoh}^{q}_{\mathcal{N}}(\mathrm{Loc}_G(\Sigma))$$

Litmus test: $Z^{(?)}_{G,\Psi}(\Sigma \times S^1) \simeq HH_*(Z^{(?)}_{G,\Psi}(\Sigma))$