DERIVING GAUGE-STRING DUALITY

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ROADMAP

- What does it mean to derive AdS/CFT? Why is it important?
- Sketch of a Program ("*l'Esquisse d'un Programme*"): From worldlines to worldsheets.
- Proof of Concept - AdS_3/CFT_2 Correspondence in the tensionless limit:

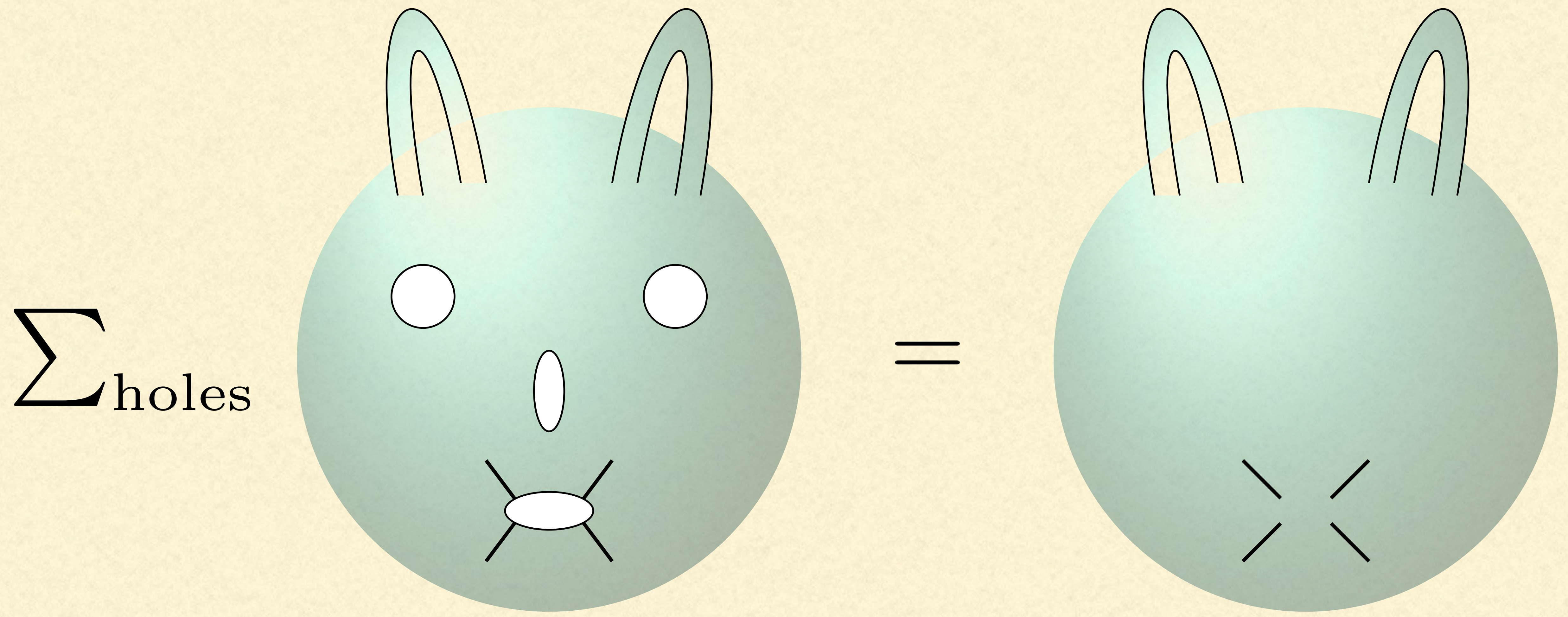
  A) From Fields to Strings

  B) From Strings to Fields

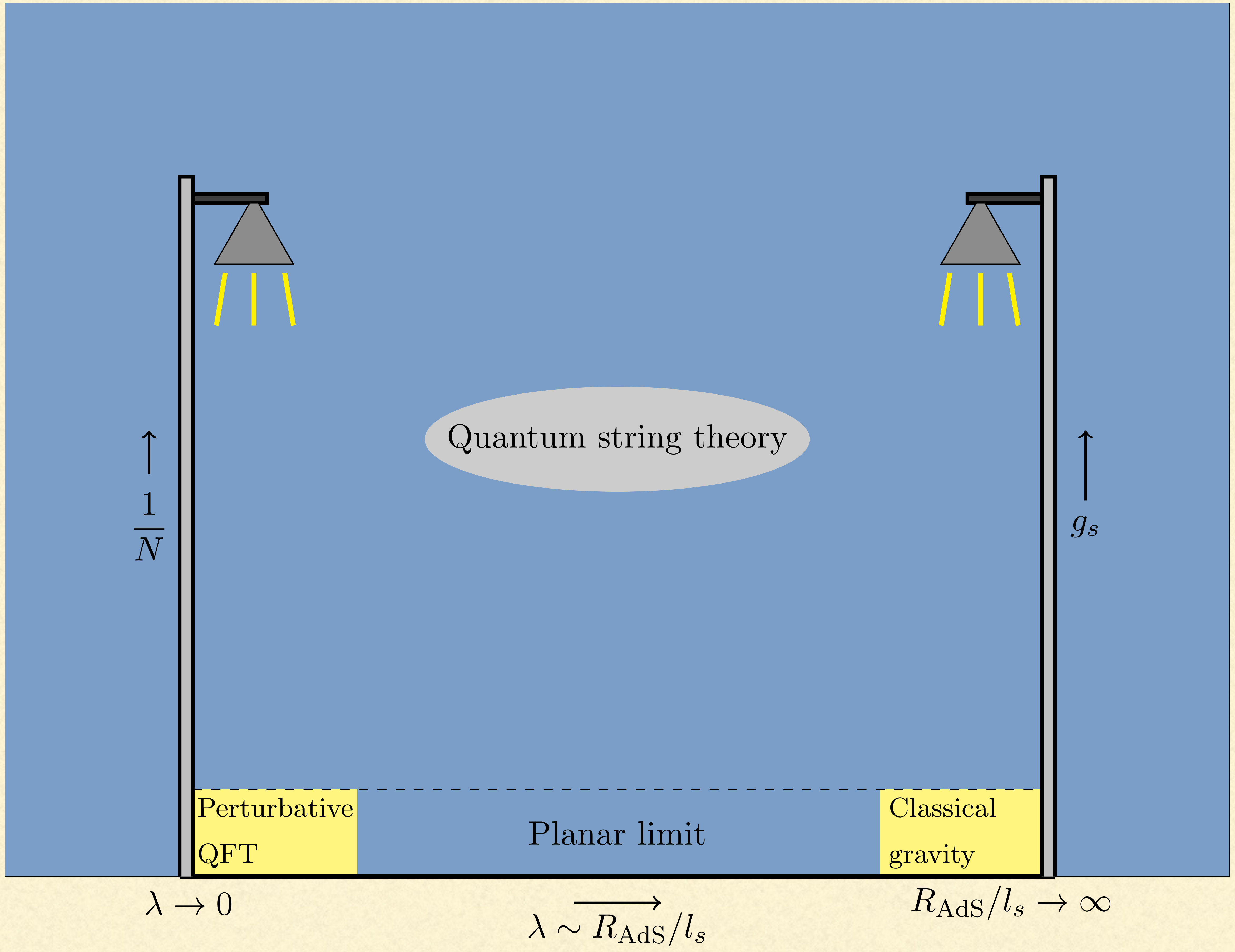
- Looking Ahead (to $\mathcal{N} = 4$ Super Yang-Mills, ...)

DERIVING AdS/CFT

- How exactly do large N QFTs reorganise themselves into theories of strings? ['t Hooft -'74]
- D-brane physics indicates open-closed string duality as the underlying reason [Maldacena-'97].
- Holes close up and backreaction alters the background.
- But difficult to see this explicitly happen at large $g_s N = \lambda$.
- Therefore, cannot delineate scope of gauge-string duality beyond examples.



A DIFFERENT LAMP POST



- Shift focus to the corner where we understand the field theory but not necessarily the bulk.
- Look at $\lambda \to 0$ i.e. highly curved AdS or tensionless limit. Very stringy regime. [cf. Sundborg, Sezgin-Sundell, …]
- Dictionary: $R_{AdS} \propto \lambda^{\alpha}$; $g_s^2 \propto \frac{\lambda^2}{N^2}$.
- Finite number of holes to sum over at zero coupling. Well defined genus expansion.
- Interactions treated perturbatively (free correlators).
- Test cases: $(T^4)^N/S_N \leftrightarrow AdS_3$, Pert $\mathcal{N} = 4$ SYM $\leftrightarrow AdS_5$.

WHAT CONSTITUTES A DERIVATION?

$$\left\langle \mathcal{O}^{(w_1)}_{h_1}(x_1)\mathcal{O}^{(w_2)}_{h_2}(x_2)\ldots\mathcal{O}^{(w_n)}_{h_n}(x_n)\right\rangle_{S^d}\Bigg|_{g} = \int_{\mathcal{M}_{g,n}} \left\langle \mathcal{V}^{w_1}_{h_1}(x_1;z_1)\mathcal{V}^{w_2}_{h_2}(x_2;z_2)\ldots\mathcal{V}^{w_n}_{h_n}(x_n;z_n)\right\rangle_{\Sigma_{g,n}}$$

- Operational definition: Relate (single trace) gauge invariant (euclidean) correlators to perturbative string amplitudes - $\forall(g,n)$.
- Based on the dictionary between states: $\mathcal{O}^{(w)}_{h}(x) \leftrightarrow \mathcal{V}^{w}_{h}(x;z)$. ($h =$ conformal dimension).
- Both sides have autonomous definitions: as a fixed point for QFT on LHS and in terms of a perturbative sigma model 2d CFT for the RHS. Mathematically well posed question.
- Can we make the equality manifest? Can we `tautologise' the correspondence?

SKETCH OF A PROGRAM

BACK AND FORTH

- $$\left\langle \mathcal{O}^{(w_1)}_{h_1}(x_1)\mathcal{O}^{(w_2)}_{h_2}(x_2)\ldots\mathcal{O}^{(w_n)}_{h_n}(x_n)\right\rangle_{S^d}\Big|_g = \int_{\mathcal{M}_{g,n}} \left\langle \mathcal{V}^{w_1}_{h_1}(x_1;z_1)\mathcal{V}^{w_2}_{h_2}(x_2;z_2)\ldots\mathcal{V}^{w_n}_{h_n}(x_n;z_n)\right\rangle_{\Sigma_{g,n}}$$

- An apparent asymmetry in this equality. Easier to go from RHS to LHS - Strings to Fields.

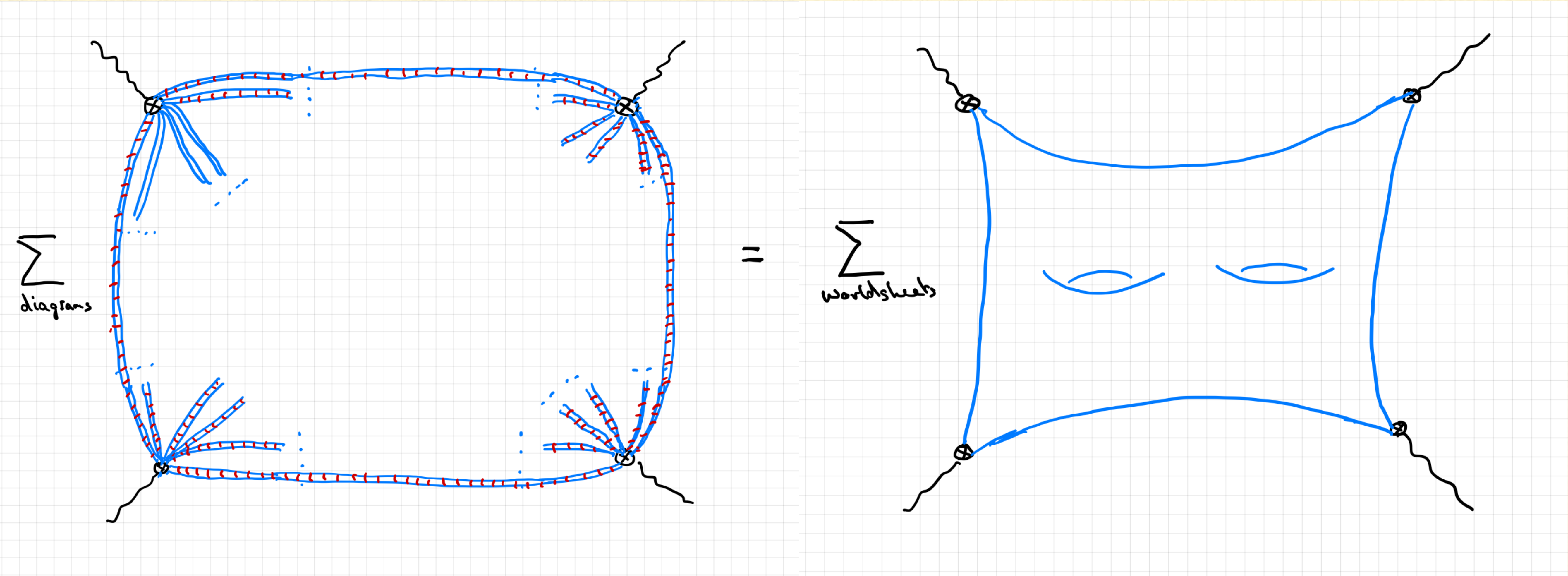
- To go from Fields to Strings (LHS to RHS), need to reconstruct a worldsheet integrand - not unique.

- Nevertheless can have a canonical or natural form for the correlator on the RHS.

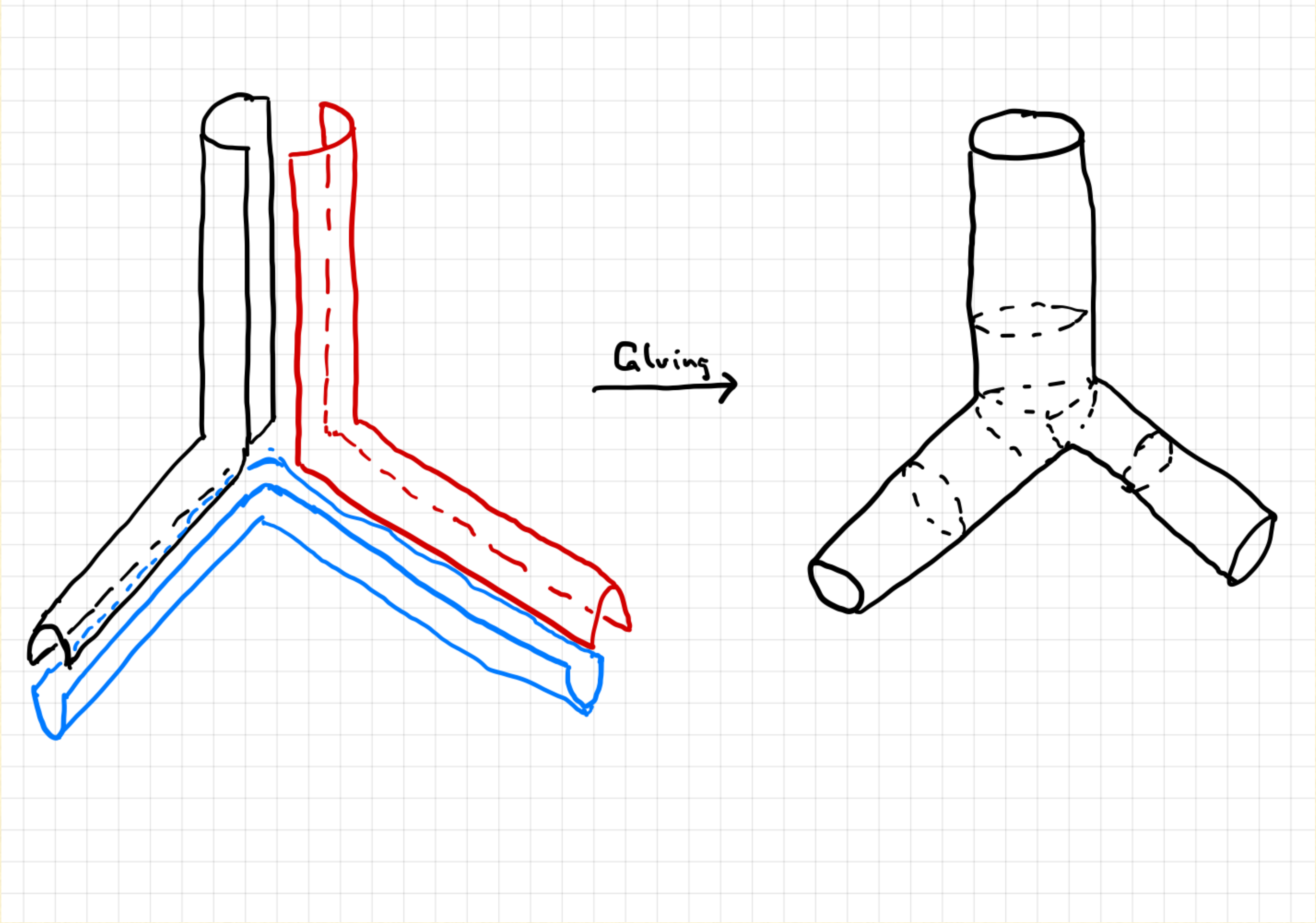
- "From Free Fields to AdS" program to recast pert. QFT correlators into stringy correlators. [R.G. '03-'05].

- BASIC IDEA: (Implicit) sum over distinct worldline topologies in Feynman diagrams for a large N theory = Sum over distinct worldsheets (moduli space) after gluing up double lines. Canonical prescription.

FROM WORLDLINES TO WORLDSHEETS



Implementation of
open-closed string duality.

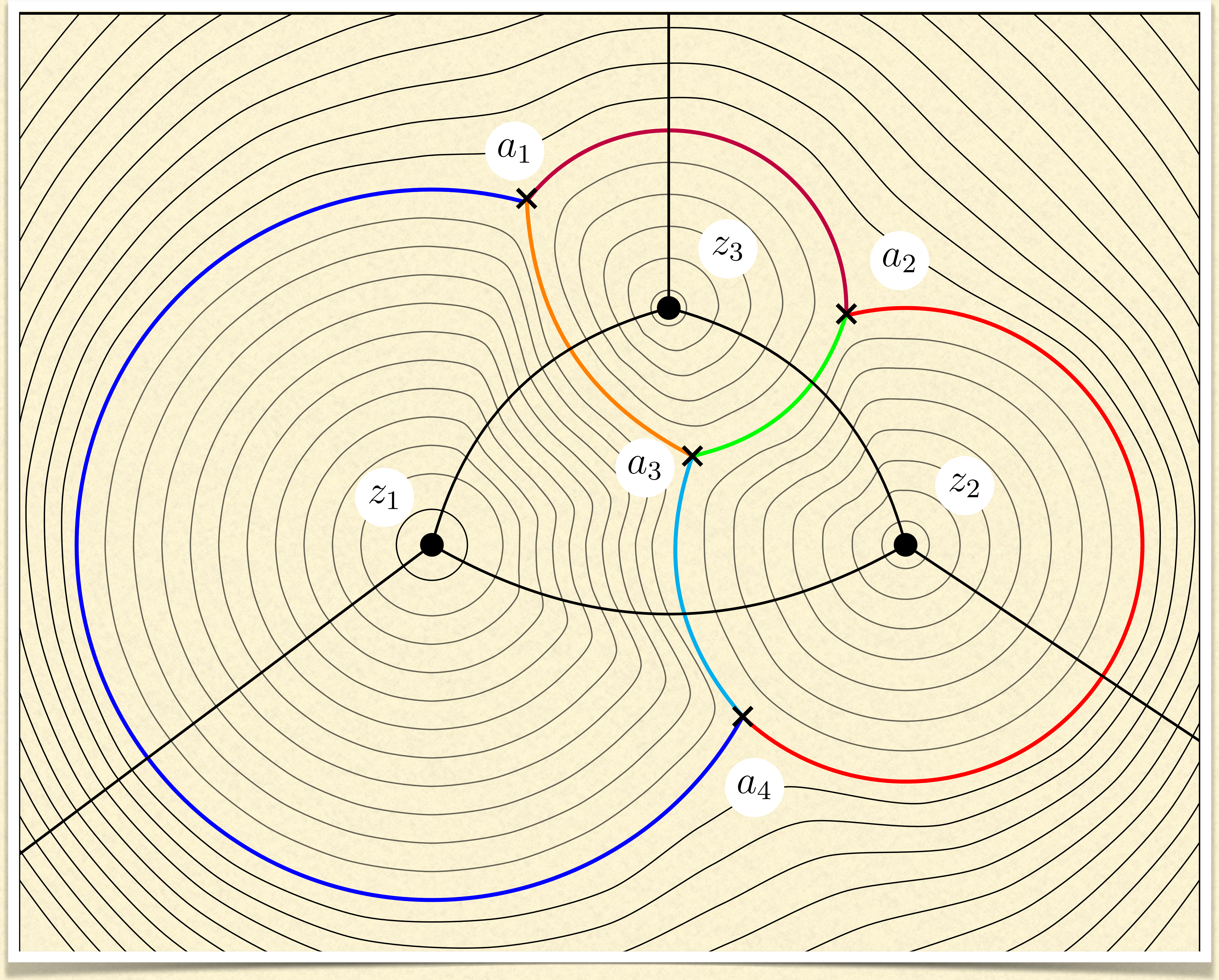


SLOGAN: EACH FEYNMAN GRAPH
$\leftrightarrow$ A CLOSED WORLDSHEET.

Exploits the Strebel parametrisation of $\mathcal{M}_{g,n}$
[R.G.'05; cf. Kontsevich'91].

- A refinement of 't Hooft's idea of associating a genus to double line Feynman graphs [R.G. '04].

STREBEL GRAPHS



- The Strebel differential foliates the Riemann surface into closed `horizontal trajectories': $\phi_S(z(t))\left(\frac{dz(t)}{dt}\right)^2 > 0.$

- Disk domains (faces) each containing one of the n double poles $\{z_i\}$.

- Separated by a critical graph connecting the zeroes $\{a_k\}$- the dual to the (skeleton) graph for the Feynman diagram.

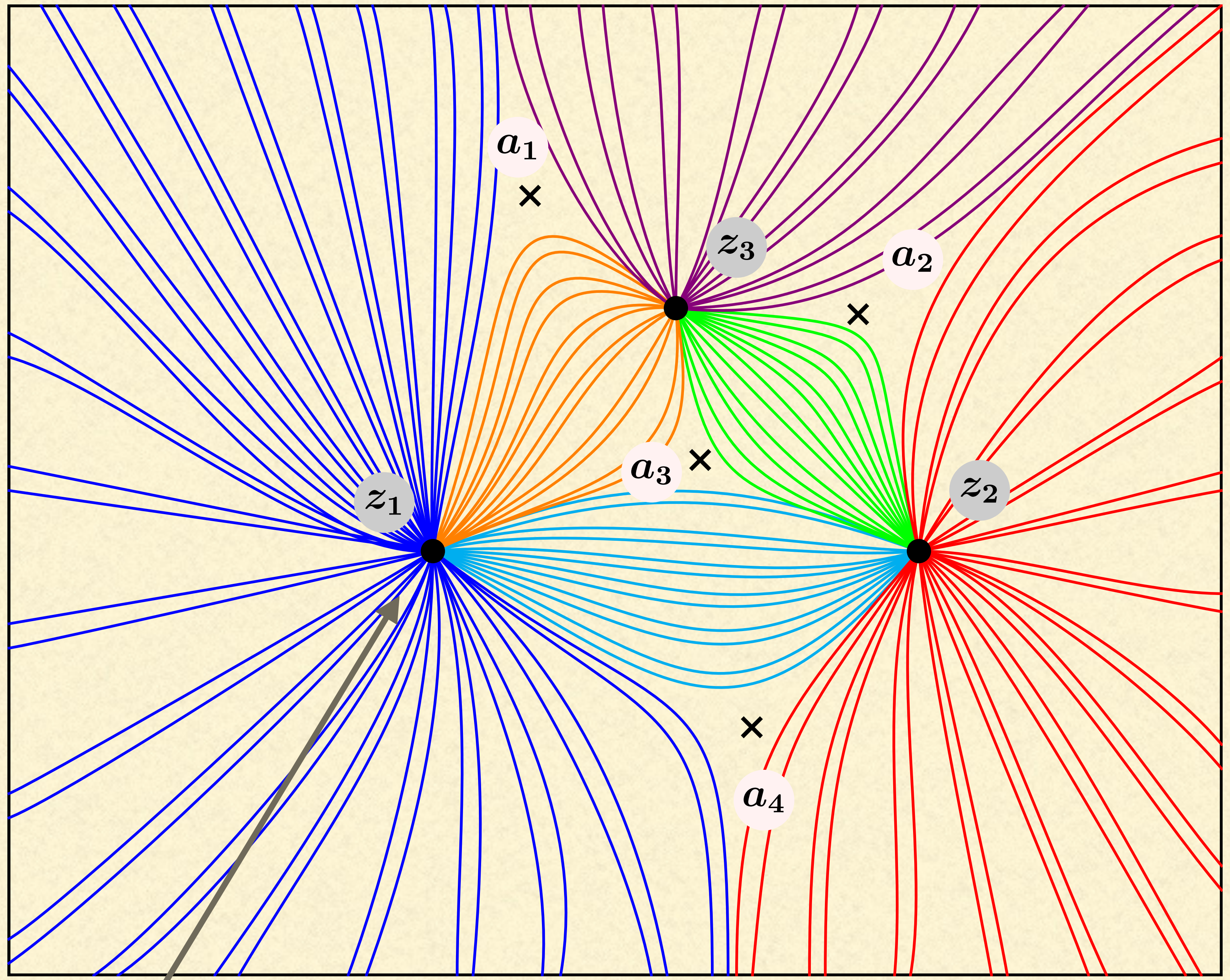
- Strebel lengths $l_{km} = \int_{a_k}^{a_m} \sqrt{\phi_S(z)}dz \propto n_{km}$ = # of Wick contractions (Razamat'08) i.e. $\in \mathbb{Z}_+$.

  Parametrise $\mathscr{M}_{g,n} \times \mathbb{R}_+^n$

- Correlators localised to discrete (integral) points on $\mathscr{M}_{g,n}$. Connection to Dessins d'enfant (Mulase-Penkava).

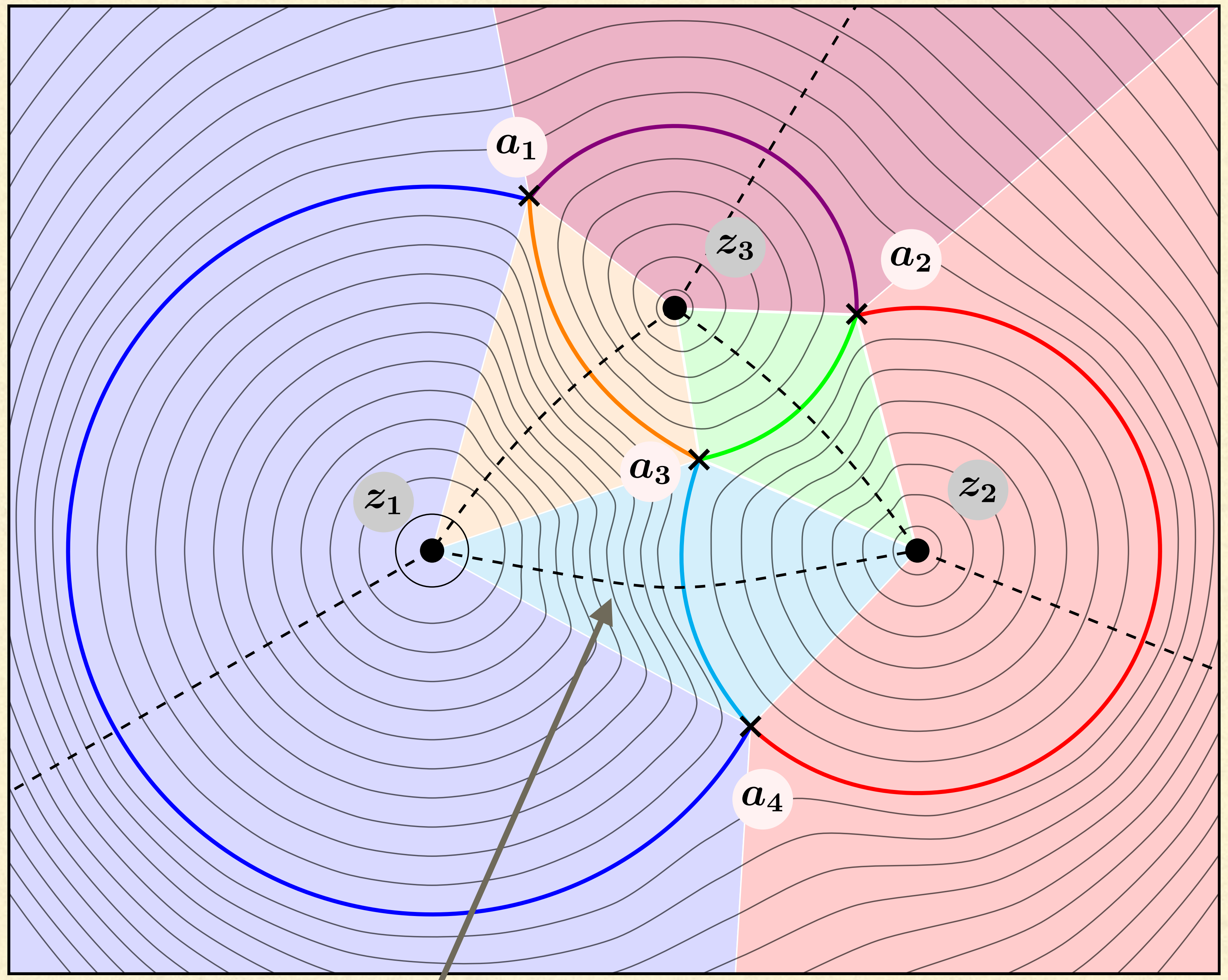
OPEN-CLOSED TRIPTYCH

1) Ribbon Graphs



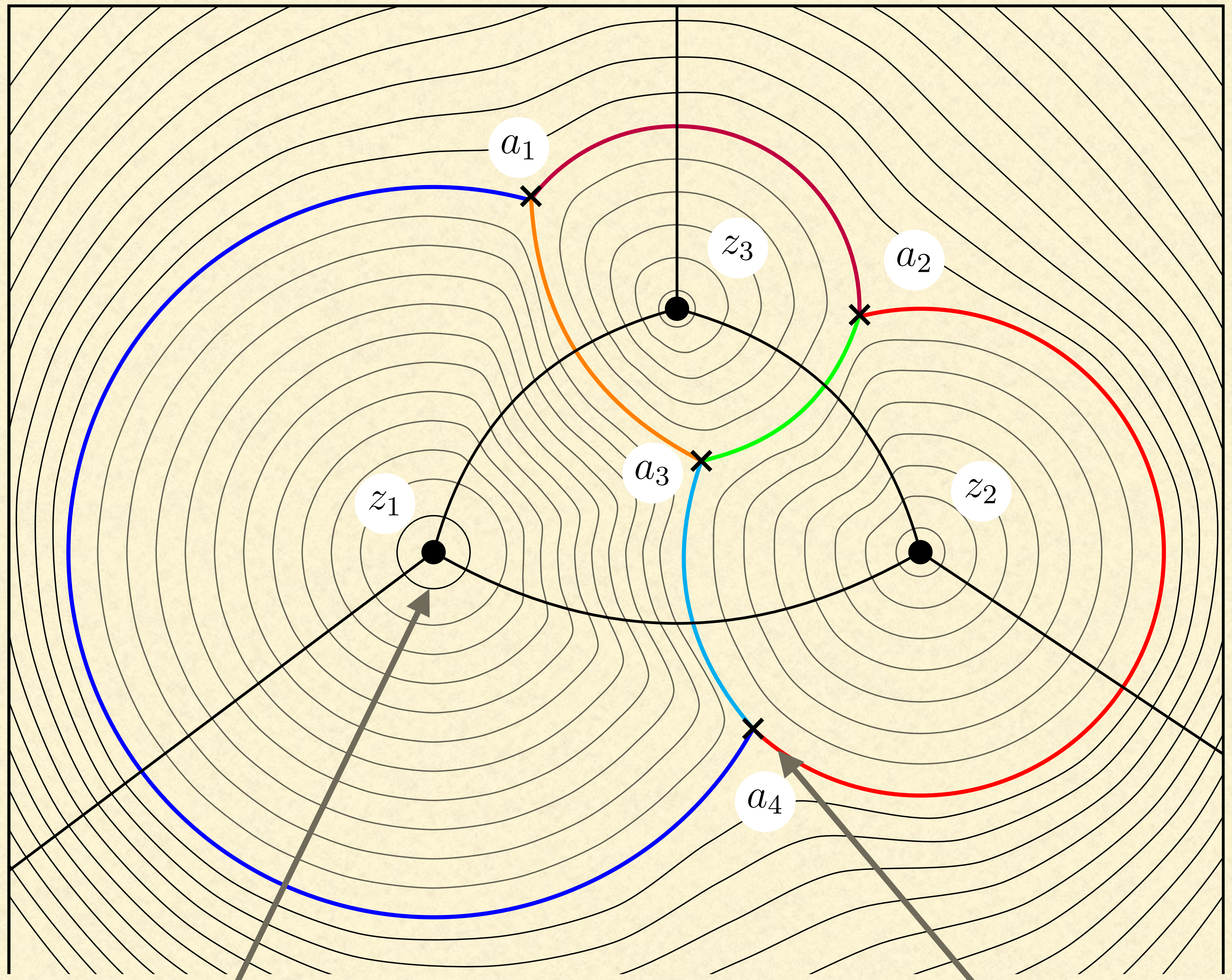
Operator Insertions

2) Glued up Strips



Conformal Strip

3) Strebel Surface



Pole

Zero

PROOF OF CONCEPT

THE AdS_3/CFT_2 CORRESPONDENCE

- Tensionless limit of the AdS_3/CFT_2 correspondence makes much of this discussion very concrete and explicit - can carry through this program

$$\left\langle \sigma_{h_1}^{(w_1)}(x_1)\sigma_{h_2}^{(w_2)}(x_2)\ldots\sigma_{h_n}^{(w_n)}(x_n)\right\rangle_{S^2}\Bigg|_g = \int_{\mathcal{M}_{g,n}} \left\langle \mathcal{V}_{h_1}^{w_1}(x_1;z_1)\mathcal{V}_{h_2}^{w_2}(x_2;z_2)\ldots\mathcal{V}_{h_n}^{w_n}(x_n;z_n)\right\rangle_{\Sigma_{g,n}}$$

- CLAIM: String Theory on $AdS_3 \times S^3 \times T^4$ and $k=1$ unit of NS-NS flux $\equiv Sym^N(T^4)$ free Symmetric Orbifold CFT as $N \to \infty$; $(g_s^2 \propto 1/N)$. [Eberhardt, Gaberdiel, R.G. - '18-'19].

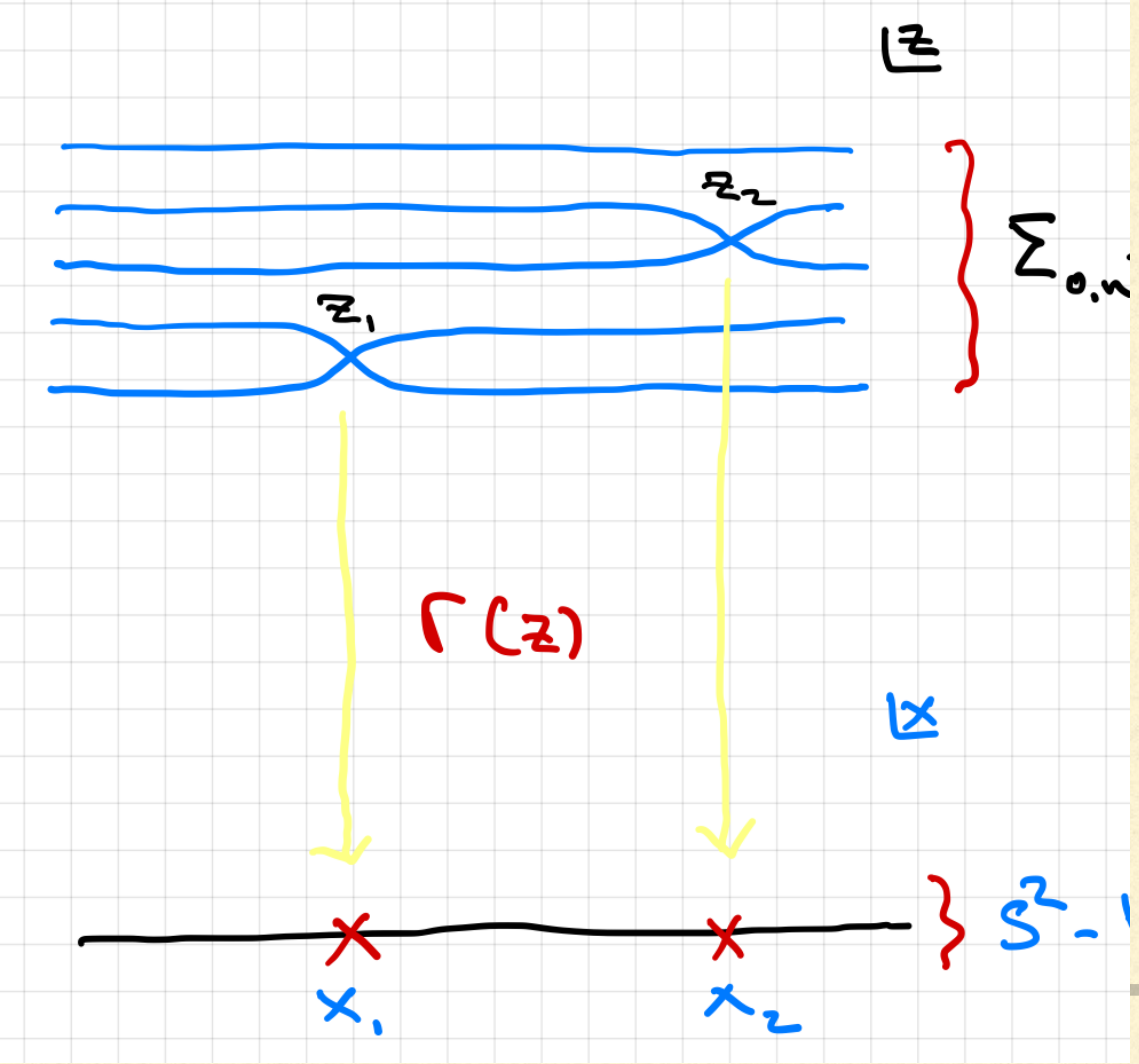
- Will be able to go from LHS to RHS using the ideas sketched earlier (explicitly for large w_i).

- Also go from RHS to LHS using unusual properties of the worldsheet CFT at $k=1$.

A. FROM FIELDS TO STRINGS

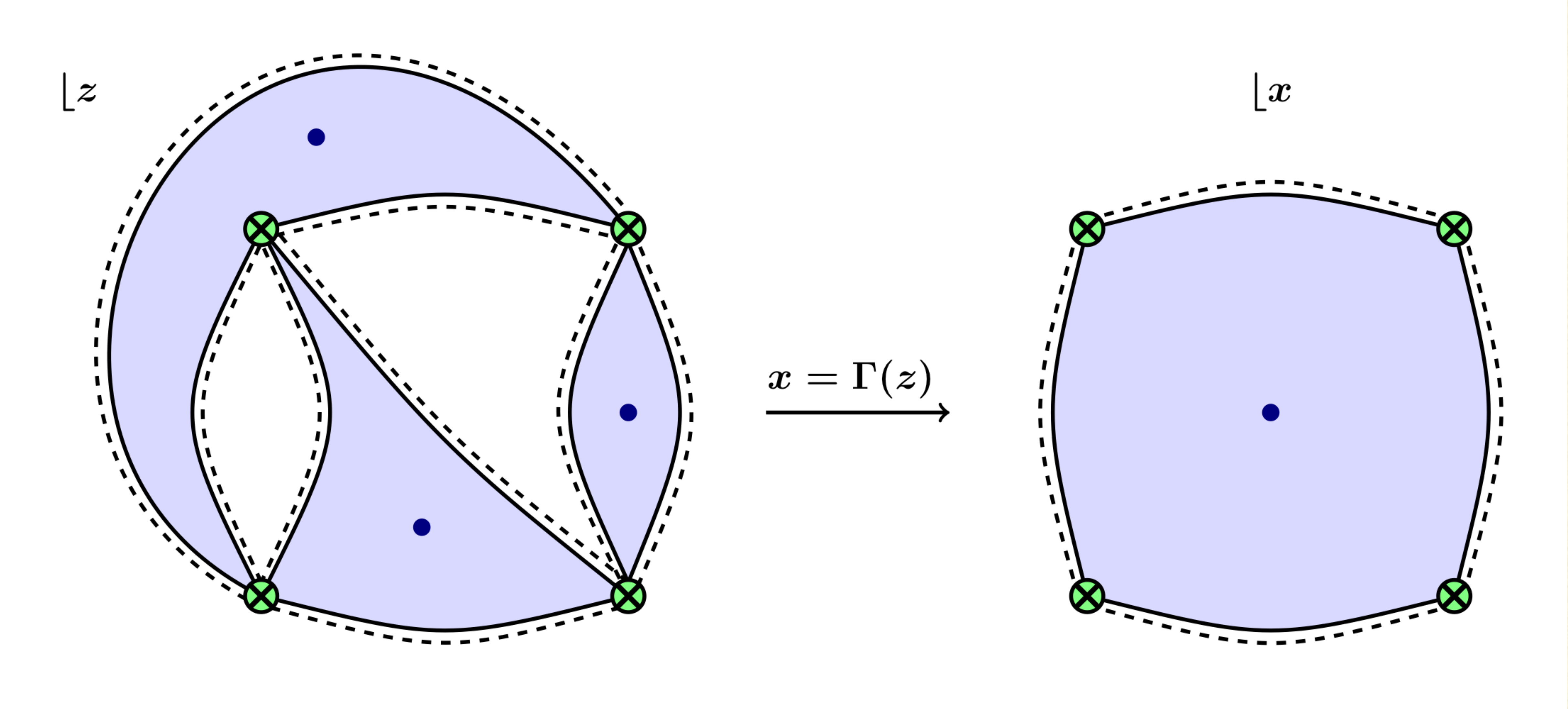
ORBIFOLD CORRELATORS AND COVERINGS

- Implement the Fields to Strings program in our test case. $CFT_2 = (T^4)^K / S_K;\ (K \to \infty)$.
- Consider $\left\langle \sigma^{(w_1)}(x_1)\sigma^{(w_2)}(x_2)\ldots\sigma^{(w_n)}(x_n)\right\rangle_{S^2}$ - ground states of w-cycle twisted sector.
- Lunin-Mathur['00] : compute by going to covering space.
- Vacuum path integral ($\sigma^{(w)}(x) \to \mathbf{1}$) of single copy of T^4 CFT.
- Locally, $x = \Gamma(z)$ with branching w_i at insertions z_i :
  $x \sim x_i + a_i^\Gamma (z - z_i)^{w_i}$. Globally, rigid problem: z_i fixed by (x_i, w_i).
- Coordinate dependence comes from pullback $\partial\Gamma(z)$ and Liouville action.
  Weight $\propto e^{-S_L[\ln|\partial\Gamma|^2]}$. $S_L[\Phi] = \dfrac{c}{48\pi}\displaystyle\int d^2z[2\partial\Phi\bar{\partial}\Phi + R\Phi]$.



FEYNMAN COVERINGS

- Can associate a free field like Feynman diagram with each contribution to symm. orbifold correlators.
- Bifundamental like double line graph - pullback of Jordan curve on spacetime S^2.
- $2w_i$ edges coming out of vertices $z_i = \Gamma^{-1}(x_i)$.
- N preimages of $x = \infty$ (poles of $\Gamma(z)$) in the coloured loops. $\left(N = 1 + \sum_{i=1}^{n} \frac{w_i - 1}{2}\right.$, Riemann-Hurwitz$\left.\right)$
- Graph triangulates the covering space = worldsheet.
- Each covering map from a distinct point on the moduli space.



[Pakman-Rastelli-Razamat-'09]

COVERING MAPS & A MATRIX MODEL

- Covering maps are hard to explicitly write down - even for genus zero.

$$\Gamma(z) = \frac{p_N(z)}{q_N(z)} = \frac{p_N(z)}{\prod_{a=1}^{N}(z-\lambda_a)} \Rightarrow \qquad \partial\Gamma(z) = M_\Gamma \frac{\prod_{i=1}^{n-1}(z-z_i)^{w_i-1}}{\prod_{a=1}^{N}(z-\lambda_a)^2}$$

[Roumpedakis -'18]

- Requiring no simple pole at $z=\lambda_a$ $\Rightarrow \sum_{i=1}^{n-1}\frac{w_i-1}{\lambda_a-z_i} = \sum_{b\neq a}^{N}\frac{2}{\lambda_a-\lambda_b}\,, \qquad (a=1,\ldots,N).$

- Simplification at large N. Saddle point of a Penner-like matrix model with potential

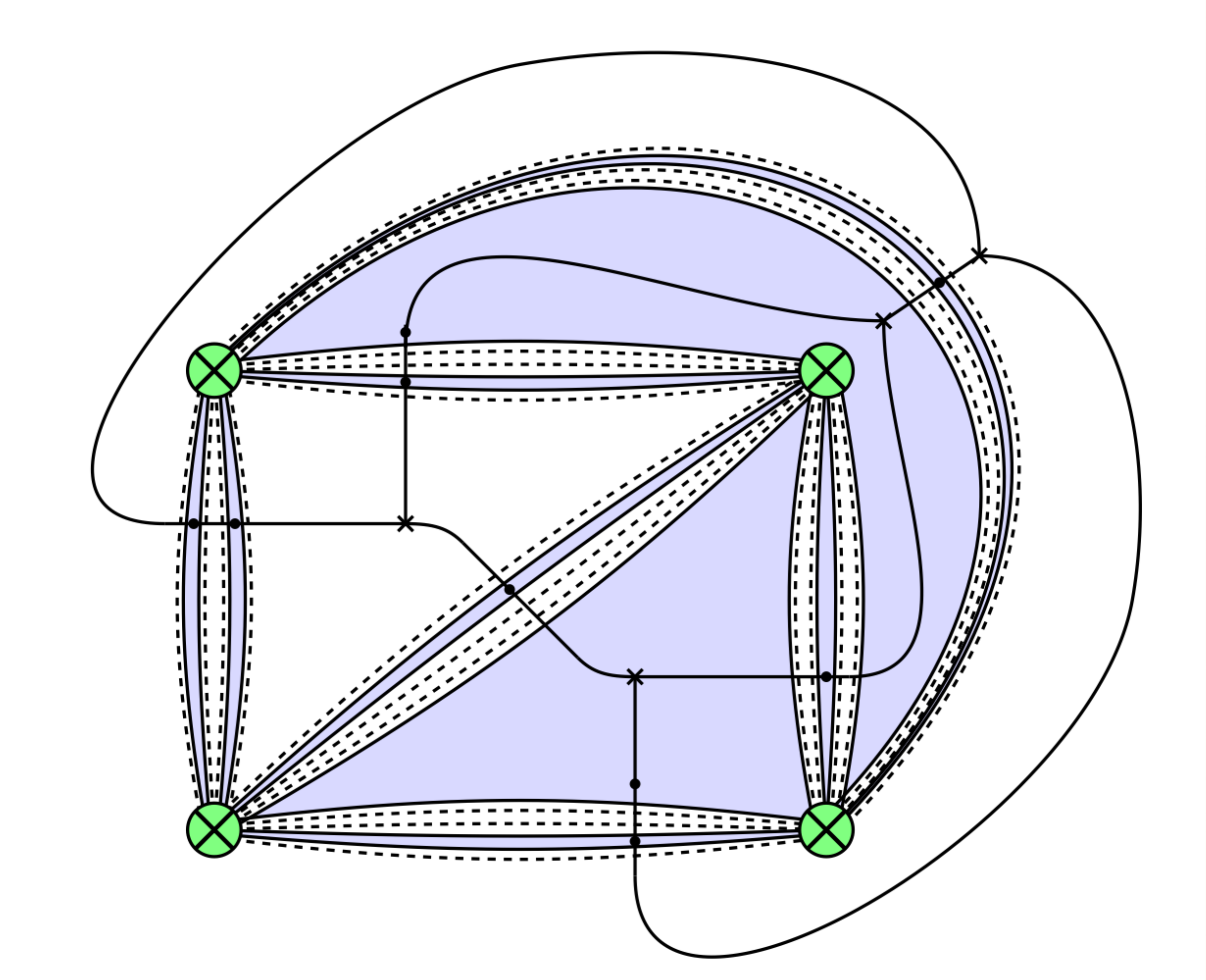
$W(z) = \sum_{i=1}^{n-1}\alpha_i \log(z-z_i)$. Introduce $\rho(\lambda) = \frac{1}{N}\sum_{a=1}^{N}\delta(\lambda-\lambda_a)$. Resolvent $u(z) = \sum_{a=1}^{N}\frac{1}{z-\lambda_a}$.

$(\alpha_i = \frac{w_i}{N})$

[Gaberdiel-R.G.-Knighton-Maity - '20]

MATRIX MODEL & FEYNMAN DIAGRAMS

- The solution of the large N matrix model encoded in a spectral curve $y_0(z) = W'(z) - 2u(z)$ - determines 'eigenvalue density' of poles λ_a in coloured loops.
- Coalesces into cuts - transverse to the edges. Forms the dual to the skeleton graph to the original graph.



- $y_0^2(z) = \dfrac{Q_{2n-4}(z)}{\prod_{i=1}^{n-1}(z-z_i)^2} = \dfrac{\alpha_n^2\, dz^2}{\prod_{i=1}^{n}(z-z_i)^2}\prod_{k=1}^{2n-4}(z-a_k)$
- Periods=`Filling fractions':

$$\frac{1}{2\pi i}\oint_{A_l} y_0(z)dz \equiv \nu_l = \frac{n^{(l)}}{N}\ , \quad \frac{1}{2\pi i}\oint_{B_l} y_0(z)dz \equiv \mu_l = \frac{\tilde{n}^{(l)}}{N}$$

STREBEL APPEARS!

- The spectral curve differential is a Strebel differential! $\phi_S(z)dz^2 = -y_0^2(z)dz^2$.

- $(2n-6)$ real periods $\sim \dfrac{n_{ij}}{N}$ take arbitrary real values (as $N \to \infty$) and parametrise the solution to the covering maps. But now see that it (Strebel) parametrises the (arithmetic) points on $\mathcal{M}_{0,n}$.

- As $N \to \infty$, the sum goes over to an integral over moduli space $\mathcal{M}_{0,n}$.

- Realises the program of associating Feynman diagrams to points in moduli space (via Strebel).

- Integrand on moduli space $\propto e^{-N^2\mathcal{S}_{cl}[\Gamma]}$. With $\mathcal{S}_{cl}[\Gamma] \propto \int d^2z\,|S[\Gamma]|$. (From Liouville action).

- $S[\Gamma] = \dfrac{\Gamma'''}{\Gamma'} - \dfrac{3}{2}\left(\dfrac{\Gamma''}{\Gamma'}\right)^2$ - the Schwarzian of the covering map - also equals $\phi_S(z)$ at large N !

B. FROM STRINGS TO FIELDS

TENSIONLESS STRINGS ON AdS_3

- Novel features of the worldsheet theory in the tensionless limit:

1. Free field (GLSM) description $[\mathfrak{psu}(1,1|2)_1]$ despite being highly curved AdS_3 - like $\mathfrak{su}(2)_1$.
2. In terms of holomorphic twistor variables: 2 symplectic bosons ($\xi^{\pm}$) and 2 fermions (+ conjugates).
3. Spectrally flowed sectors $\{w_i\}$ of the WZW model $\leftrightarrow$ twisted sectors $\{w_i\}$ of dual orbifold CFT.
4. Worldsheet correlators of these sectors are delta function localised to discrete points on $\mathscr{M}_{g,n}$.
5. Semiclassical worldsheet which is essentially at the boundary of AdS_3 - gives Lunin-Mathur correlators.

FROM STRINGS TO FIELDS: SPECTRUM

- The entire (not just BPS) spectrum of the perturbative string theory exactly matches with the (single cycle) states of the large N 2d orbifold CFT. [Eberhardt, Gaberdiel, R.G. -'18]

$$\mathscr{V}_h^w(x;z) \quad \longleftrightarrow \quad \mathscr{O}_h^{(w)}(x)$$

- h = spacetime conformal dimension = AdS_3 energy. w = twisted sector cycle = spectral flow.
- The bulk theory with $k=1$ has fewer states than for $k>1$. No continuum of long strings.
- Only $j=1/2$ multiplets under $\mathfrak{sl}_2(\mathbb{R})$ due to truncation in $\mathfrak{psu}(1,1|2)_1$ WZW model.
- Only four transverse oscillators (T^4); Quasi-topological on $AdS_3 \times S^3$.

FROM STRINGS TO FIELDS: CORRELATORS

$$\left\langle \sigma^{(w_1)}(x_1)\sigma^{(w_2)}(x_2)\ldots\sigma^{(w_n)}(x_n)\right\rangle_{S^2}\bigg|_{g=0} = \int_{\mathcal{M}_{0,n}} \left\langle \mathcal{V}_0^{w_1}(x_1;z_1)\mathcal{V}_0^{w_2}(x_2;z_2)\ldots\mathcal{V}_0^{w_n}(x_n;z_n)\right\rangle_{\Sigma_{0,n}}$$

- Restrict to ground states in w-twisted sectors; and genus zero - for simplicity. RHS nontrivially agrees with LHS. (See generalisation to BPS and other states - Gaberdiel-Nairz '22)

- Because of unusual localisation of worldsheet correlators on moduli space - to holomorphic maps $x = \Gamma(z)$ with branching w_i at insertions z_i: $x \sim x_i + a_i^{\Gamma}(z-z_i)^{w_i}\,;(i=1,2\ldots,n)$.

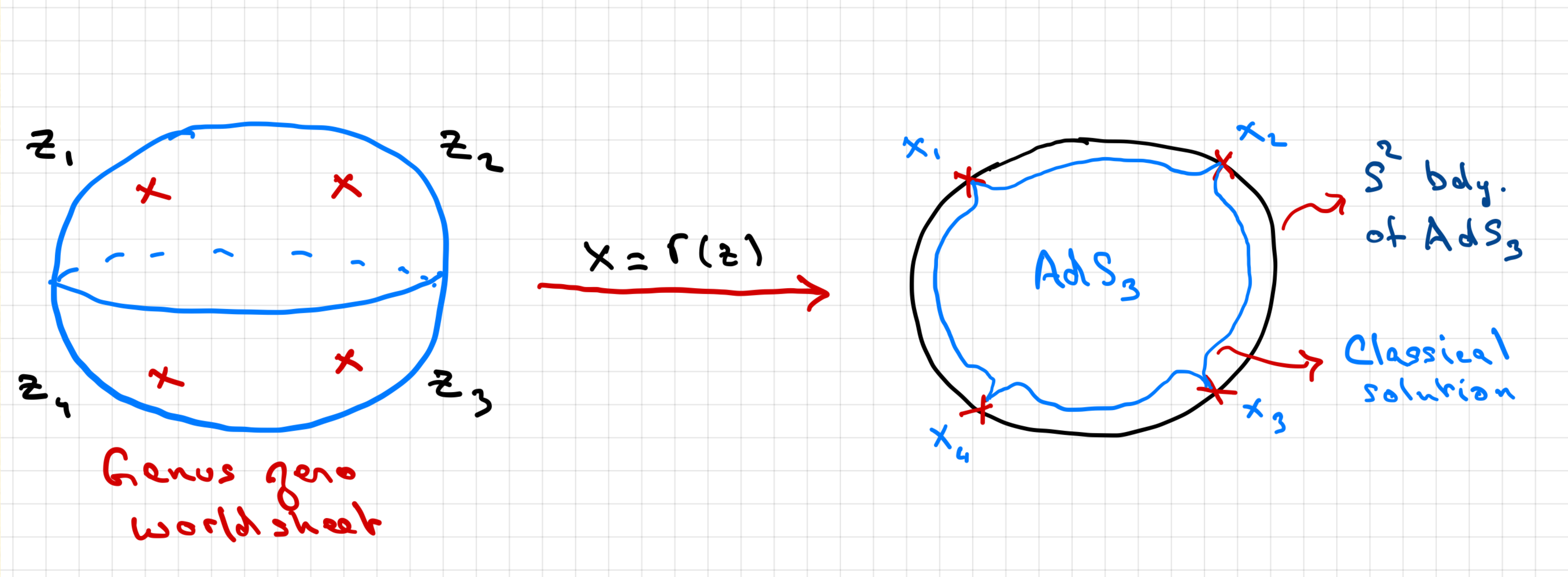
  [Cf. Eberhardt @StringMath21]

- CLAIM: Worldsheet correlator on RHS $\propto \prod_{i=4}^{n}\delta^{(2)}(x_i-\Gamma(z_i))$ - discrete set of points allowing covers.

  [Eberhardt-Gaberdiel-R.G. -'19]

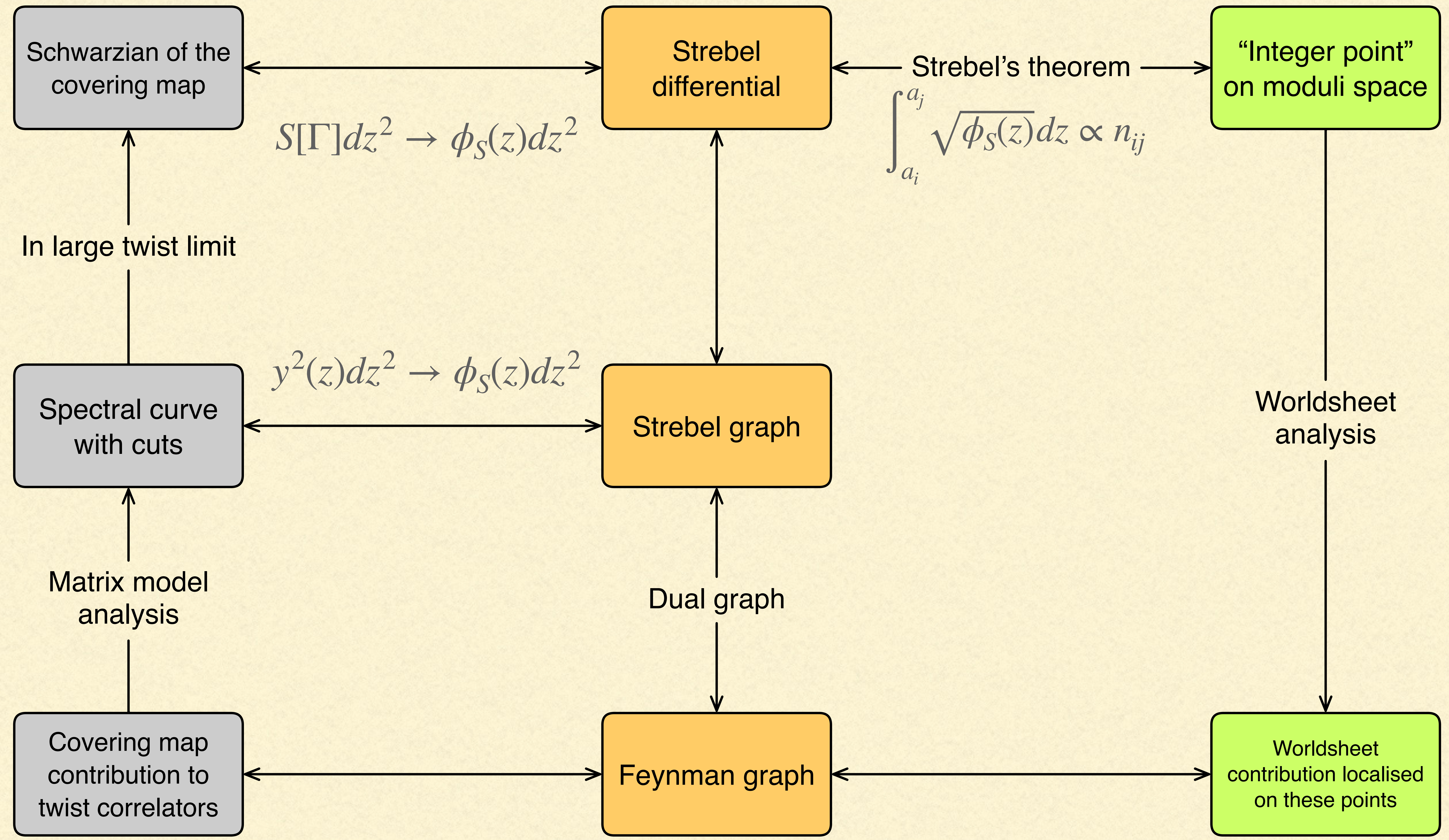
- Exact semiclassical worldsheet sigma model action gives weight $\propto e^{-S_L[\Phi=\ln|\partial\Gamma|^2]}$, $\Phi =$ radial coord.

TWISTORS & LOCALISATION



- Thus worldsheet is a covering space of the boundary S^2 exactly as in Lunin-Mathur.
- This localisation is transparent in a free field realisation of $\mathfrak{psu}(1,1|2)_1$ - twistor variables $Z^I = (\xi^\alpha, \psi^\alpha); Y_I = (\epsilon_{\alpha\beta}\eta^\beta, \epsilon_{\alpha\beta}\chi^\beta)$. [Dei, Gaberdiel R.G., Knighton- '20]
- Twistor incidence relation: $\langle(\xi^-(z) + \Gamma(z)\xi^+(z))\rangle_{phys} = 0$
- Implies that correlators are $\propto \sum_\Gamma \hat{W}_\Gamma \prod_{i=1}^n |a_i^\Gamma|^{-2h_i} \prod_{i=4}^n \delta^{(2)}(x_i - \Gamma(z_i))$

LOGIC FLOWCHART



LOOKING AHEAD

TENSIONLESS STRINGS ON AdS_5

Twistorial Gauged Linear Sigma Model for
$AdS_3 \times S^3 : Y_I = (\eta_\alpha, \chi_\beta);\ Z^I = (\xi^\alpha, \psi^\beta)\,.$

Twistorial Gauged Linear Sigma Model for
$AdS_5 \times S^5 : Y_I = (\mu^\dagger_\alpha, \lambda^\dagger_{\dot\alpha}, \psi^\dagger_a);\ Z^I = (\lambda^\alpha, \mu^{\dot\alpha}, \psi^a)\,.$
[Gaberdiel-R. G. '21]

Ambitwistor Open String Theory (Y_I, Z^I)
[Berkovits'04; Mason-Skinner'13….]

BMN & Integrable Spin Chains
[Berenstein-Maldacena-Nastase '02,….]

FREE FIELDS ON THE WORLDSHEET

- Twistor fields $Z^I = (\lambda^\alpha, \mu^{\dot\beta}, \psi^a)$; $Y_J = (\lambda^\dagger_{\dot\alpha}, \mu^\dagger_\beta, \psi^\dagger_b)$ give a free field representation of $\mathfrak{psu}(2,2|4)$, through bilinears $Y_I Z^J$ (with $\mathscr{C} \equiv Y^I Z_I = 0$ - projects out the $\mathfrak{u}(1)$).

- For each $w \in \mathbb{Z}_+$ consider Fock space built on $|0\rangle_w$ by a finite number of "wedge modes":

$$(\mu^\dagger_\alpha)_r\ , \quad (\mu^{\dot\alpha})_r\ , \quad (\psi^\dagger_a)_r \ \ (a = 1,2)\ , \quad \psi^b_r \ \ (b = 3,4)\ , \quad \text{with} \quad -\tfrac{w-1}{2} \le r \le \tfrac{w-1}{2}$$

- $|0\rangle_w$ is a "spectrally flowed" vacuum state.

- $w = 0 \leftrightarrow$ NS sector, $w = 1 \leftrightarrow$ Ramond sector: only zero modes. These generate the singleton representation of the 4d superconformal algebra $\mathfrak{psu}(2,2|4)$.

PHYSICAL GAUGE AND SPECTRUM

- PROPOSAL: In a "physical gauge" can gauge away out-of-the-wedge modes leaving only the wedge modes $Z^I_r, (Y_J)_r$ (with $-\frac{(w-1)}{2} \leq r \leq \frac{(w-1)}{2}$) - left with w bits. Underlying worldsheet $\mathcal{N} = 4$?

[Gaberdiel-R.G. '21]

- Further, impose residual physical state conditions

$$\text{A)}\ (L_0 + pw)\,|phys\rangle_w = 0 \quad (p \in \mathbb{Z})$$

$$\text{B)}\ \mathscr{C}_r\,|phys\rangle_w = 0 \quad (r = 0,1,\ldots(w-1)) \qquad [\mathscr{C} = Z^I Y_I]$$

- Condition A $\leftrightarrow$ discrete translation invariance on a worldsheet with w bits (cyclicity).

- Condition B $\leftrightarrow$ Each of the individual bits at each "site" form a singleton.

$$\hat{\Phi}_j = \frac{1}{\sqrt{w}} \sum_{r=-\frac{w-1}{2}}^{\frac{w-1}{2}} \Phi_r e^{i2\pi jr/w}$$

Any Twistor Field

FREE $\mathcal{N} = 4$ SYM FROM THE WORLDSHEET

- Reproduces precisely the large N gauge invariant spectrum of single trace operators (w letters) in $\mathcal{N} = 4$ SYM : $\sum_w (singleton)^{\otimes w} / (cyclicity)$. [$w = 0 \leftrightarrow \mathbf{1}$ (identity operator in SYM)]

  [Bianchi, Morales, Samtleben; Alday, David, Gava, Narain]

- Singleton $(w = 1) \leftrightarrow$ single "letters" of SYM $\{\partial^s \phi^i, \partial^s \Psi^a_\alpha, \partial^s \bar{\Psi}^{\dot{\alpha}}_a, \partial^s \mathcal{F}_{\alpha\beta}, \partial^s \mathcal{F}^{\dot{\alpha}\dot{\beta}}\}$.

- Gives an organisation of the free SYM spectrum in terms of w bits - same building blocks of the integrable spin chains that govern the dynamics of perturbative SYM [Minahan-Zarembo; Beisert, Staudacher et.al].

- Need worldsheet quantisation that reproduces these physical modes and residual constraints.

  [Gaberdiel, R. G. (In progress)]

- Expect a localisation of correlators to holomorphic maps into ambitwistor space. $S[\Gamma] \propto \phi_s \Rightarrow e^{-2\pi A_S} \propto \left(\frac{1}{x_{ij}^2}\right)^w$

  — Feynman Propagators! [Bhat, Maity, R. G., Radhakrishnan '22]

OUTLOOK

- Exhibited a test case (tensionless AdS_3/CFT_2) where the general program of reassembling large N QFTs into string theories can be explicitly carried out.

- Could also close the circle from strings to fields due to a tractable worldsheet theory.

- Extension to perturbative $\mathcal{N} = 4$ SYM in terms of twistor description of AdS_5. Compelling ingredients for worldsheet theory but needs to be put on solid footing.

- Extend "Fields to Strings" program to other large N QFTs. E.g. string duals to large N Matrix models [R.G.-Mazenc '22, in progress].

- Tensionless string theories on AdS likely to give a new, unusual family of topological string theories. Interesting connections to Mathematics.

THANK YOU