DERIVING GAUGE-STRING DUALITY

Collaboration with: Matthias Gaberdiel
Also P. Maity, B. Knighton, A. Dei, F. Bhat, R. Bharathkumar

Rajesh Gopakumar,
ICTS-TIFR, Bengaluru.

StringMath22 (Warsaw),
ROADMAP

- What does it mean to derive AdS/CFT? Why is it important?
- Sketch of a Program (“l’Esquisse d’un Programme”): From worldlines to worldsheets.
- Proof of Concept - $AdS_3/CFT_2$ Correspondence in the tensionless limit:
  A) From Fields to Strings
  B) From Strings to Fields
- Looking Ahead (to $\mathcal{N} = 4$ Super Yang-Mills, …..)
DERIVING AdS/CFT

- How exactly do large N QFTs reorganise themselves into theories of strings? ['t Hooft -'74]
- D-brane physics indicates open-closed string duality as the underlying reason [Maldacena-'97].
- Holes close up and backreaction alters the background.
- But difficult to see this explicitly happen at large $g_s N = \lambda$.
- Therefore, cannot delineate scope of gauge-string duality beyond examples.
A DIFFERENT LAMP POST

- Shift focus to the corner where we understand the field theory but not necessarily the bulk.
- Look at $\lambda \to 0$ i.e. highly curved AdS or tensionless limit. Very stringy regime. [cf. Sundborg, Sezgin-Sundell, …]
- Dictionary: $R_{\text{AdS}} \propto \lambda^\alpha$; $g_s^2 \propto \frac{\lambda^2}{N^2}$.
- Finite number of holes to sum over at zero coupling. Well defined genus expansion.
- Interactions treated perturbatively (free correlators).
- Test cases: $(T^4)^N/S_N \leftrightarrow \text{AdS}_3$,
  Pert $\mathcal{N} = 4$ SYM $\leftrightarrow \text{AdS}_5$. 

Figure 1. The lamp post figure

Figure 2. Open-closed string duality
WHAT CONSTITUTES A DERIVATION?

\[
\left\langle \mathcal{O}_{h_1}^{(w_1)}(x_1) \mathcal{O}_{h_2}^{(w_2)}(x_2) \ldots \mathcal{O}_{h_n}^{(w_n)}(x_n) \right\rangle_{S^d} = \int_{\mathcal{M}_{g,n}} \left\langle \mathcal{V}^{w_1}_{h_1}(x_1; z_1) \mathcal{V}^{w_2}_{h_2}(x_2; z_2) \ldots \mathcal{V}^{w_n}_{h_n}(x_n; z_n) \right\rangle_{\Sigma_{g,n}}
\]

- **Operational definition**: Relate (single trace) gauge invariant (euclidean) correlators to perturbative string amplitudes - \( \forall (g, n) \).

- Based on the dictionary between states: \( \mathcal{O}_h^{(w)}(x) \leftrightarrow \mathcal{V}_h^{w}(x; z) \). \((h = \text{conformal dimension})\).

- Both sides have **autonomous definitions**: as a fixed point for QFT on LHS and in terms of a perturbative sigma model 2d CFT for the RHS. Mathematically well posed question.

- Can we make the **equality manifest**? Can we `tautologise` the correspondence?
SKETCH OF A PROGRAM
BACK AND FORTH

\[
\left\langle \mathcal{O}_{h_1}^{(w_1)}(x_1)\mathcal{O}_{h_2}^{(w_2)}(x_2)\ldots\mathcal{O}_{h_n}^{(w_n)}(x_n) \right\rangle_{S^d} \bigg|_g = \int_{\mathcal{M}_{g,n}} \left\langle \mathcal{V}_{h_1}^{w_1}(x_1; z_1)\mathcal{V}_{h_2}^{w_2}(x_2; z_2)\ldots\mathcal{V}_{h_n}^{w_n}(x_n; z_n) \right\rangle_{\Sigma_{g,n}}
\]

- An apparent asymmetry in this equality. Easier to go from RHS to LHS - Strings to Fields.

- To go from Fields to Strings (LHS to RHS), need to reconstruct a worldsheet integrand - not unique.

- Nevertheless can have a canonical or natural form for the correlator on the RHS.

- “From Free Fields to AdS” program to recast pert. QFT correlators into stringy correlators. [R.G. ’03-’05].

- BASIC IDEA: (Implicit) sum over distinct worldline topologies in Feynman diagrams for a large N theory = Sum over distinct worldsheets (moduli space) after gluing up double lines. Canonical prescription.
FROM WORLDLINES TO WORLDSHEETS

SLOGAN: EACH FEYNMAN GRAPH ↔ A CLOSED WORLDSHEET.

Exploits the Strebel parametrisation of $\mathcal{M}_{g,n}$

[R.G.'05; cf. Kontsevich’91].

- A refinement of 't Hooft’s idea of associating a genus to double line Feynman graphs [R.G. ’04].
The Strebel differential foliates the Riemann surface into closed ‘horizontal trajectories’: \( \phi_S(z(t)) \left( \frac{dz(t)}{dt} \right)^2 > 0 \).

Disk domains (faces) each containing one of the \( n \) double poles \( \{ z_i \} \).

Separated by a critical graph connecting the zeroes \( \{ a_k \} \) - the dual to the (skeleton) graph for the Feynman diagram.

Strebel lengths \( l_{km} = \int_{a_k}^{a_m} \sqrt{\phi_S(z)} dz \propto n_{km} = \# \text{ of Wick contractions} \) (Razamat 08) i.e. \( \in \mathbb{Z}_+ \).

Correlators localised to discrete (integral) points on \( \mathcal{M}_{g,n} \). Connection to Dessins d’enfant (Mulase-Penkava).
Figure 2: The horizontal trajectories of a Strebel differential. The (double) poles of the Strebel differential are denoted by black dots, while the zeros are represented by black crosses. The coloured lines describe the critical horizontal trajectories that make up the critical Strebel graph, see Section ?? for more details.

1) Ribbon Graphs
2) Glued up Strips
3) Strebel Surface

Operator Insertions
Conformal Strip
Pole
Zero
PROOF OF CONCEPT
THE AdS$_3$/CFT$_2$ CORRESPONDENCE

- Tensionless limit of the AdS$_3$/CFT$_2$ correspondence makes much of this discussion very concrete and explicit - can carry through this program

$$\left\langle \sigma_{h_1}^{(w_1)}(x_1) \sigma_{h_2}^{(w_2)}(x_2) \ldots \sigma_{h_n}^{(w_n)}(x_n) \right\rangle_{S^2} / g = \int_{\mathcal{M}_{g,n}} \left\langle \mathcal{V}_{h_1}^{w_1}(x_1; z_1) \mathcal{V}_{h_2}^{w_2}(x_2; z_2) \ldots \mathcal{V}_{h_n}^{w_n}(x_n; z_n) \right\rangle_{\Sigma_{g,n}}$$

- CLAIM: String Theory on $AdS_3 \times S^3 \times T^4$ and $k = 1$ unit of NS-NS flux $\equiv Sym^N(T^4)$ free Symmetric Orbifold CFT as $N \to \infty$; $(g_s^2 \propto 1/N)$. [Eberhardt, Gaberdiel, R.G. - ’18-’19].

- Will be able to go from LHS to RHS using the ideas sketched earlier (explicitly for large $w_i$).

- Also go from RHS to LHS using unusual properties of the worldsheet CFT at $k = 1$. 
A. FROM FIELDS TO STRINGS
Implement the **Fields to Strings** program in our test case. $CFT_2 = (T^4)^K/S_K$; $(K \to \infty)$.

Consider $\langle \sigma^{(w_1)}(x_1)\sigma^{(w_2)}(x_2)\ldots\sigma^{(w_n)}(x_n) \rangle_{S^2}$ - ground states of $w$-cycle twisted sector.

Lunin-Mathur['00] : compute by going to covering space.

Vacuum path integral ($\sigma^{(w)}(x) \to 1$) of single copy of $T^4$ CFT.

Locally, $x = \Gamma(z)$ with branching $w_i$ at insertions $z_i$ : $x \sim x_i + a_i^{\Gamma}(z - z_i)^{w_i}$. Globally, rigid problem: $z_i$ fixed by $(x_i, w_i)$.

Coordinate dependence comes from pullback $\partial\Gamma(z)$ and Liouville action. Weight $\propto e^{-S_L[\ln|\partial\Gamma|^2]}$. $S_L[\Phi] = \frac{c}{48\pi} \int d^2z [2\partial\Phi\bar{\partial}\Phi + R\Phi]$. 
Can associate a free field like Feynman diagram with each contribution to symm. orbifold correlators.

**Bifundamental like double line graph** - pullback of Jordan curve on spacetime $S^2$.

2\(w_i\) edges coming out of vertices \(z_i = \Gamma^{-1}(x_i)\).

N preimages of \(x = \infty\) (poles of \(\Gamma(z)\)) in the coloured loops.

\(N = 1 + \sum_{i=1}^{P} \frac{w_i - 1}{2}\), Riemann-Hurwitz

Graph **triangulates** the covering space = worldsheet.

Each covering map from a distinct point on the moduli space.

[Pakman-Rastelli-Razamat-'09]
COVERING MAPS & A MATRIX MODEL

- **Covering maps** are hard to explicitly write down - even for genus zero.

\[
\Gamma(z) = \frac{p_N(z)}{q_N(z)} = \frac{p_N(z)}{\prod_{a=1}^{N}(z - \lambda_a)} \Rightarrow \quad \partial \Gamma(z) = M \frac{\prod_{i=1}^{n-1}(z - z_i)^{w_i-1}}{\prod_{a=1}^{N}(z - \lambda_a)^2}
\]

[Roumpedakis -'18]

- Requiring no simple pole at \( z = \lambda_a \) \( \Rightarrow \)

\[
\sum_{i=1}^{n-1} \frac{w_i - 1}{\lambda_a - z_i} = \sum_{b \neq a}^{N} \frac{2}{\lambda_a - \lambda_b}, \quad (a = 1, \ldots, N).
\]

- **Simplification at large N.** Saddle point of a Penner-like matrix model with potential

\[
W(z) = \sum_{i=1}^{n-1} \alpha_i \log (z - z_i). \quad \text{Introduce} \quad \rho(\lambda) = \frac{1}{N} \sum_{a=1}^{N} \delta(\lambda - \lambda_a). \quad \text{Resolvent} \quad u(z) = \sum_{a=1}^{N} \frac{1}{z - \lambda_a}
\]

[Gaberdiel-R.G.-Knighton-Maity - '20]
The solution of the large N matrix model encoded in a spectral curve $y_0(z) = W'(z) - 2u(z)$ - determines ‘eigenvalue density’ of poles $\lambda_a$ in coloured loops.

- **Coalesces into cuts** - transverse to the edges. Forms the dual to the skeleton graph to the original graph.

$$y_0^2(z) = \frac{Q_{2n-4}(z)}{\prod_{i=1}^{n-1} (z - z_i)^2} = \frac{\alpha_n^2 \, dz^2}{\prod_{i=1}^{n} (z - z_i)^2} \prod_{k=1}^{2n-4} (z - a_k)$$

- **Periods=‘Filling fractions’**:

$$\frac{1}{2\pi i} \oint_{A_l} y_0(z) \, dz \equiv \nu_l = \frac{n^{(l)}}{N}, \quad \frac{1}{2\pi i} \oint_{B_l} y_0(z) \, dz \equiv \mu_l = \frac{\tilde{n}^{(l)}}{N}$$
STREBEL APPEARS!

- The spectral curve differential is a Strebel differential! $\phi_S(z) dz^2 = -y_0^2(z) dz^2$.

- $(2n - 6)$ real periods $\sim \frac{n_{ij}}{N}$ take arbitrary real values (as $N \to \infty$) and parametrise the solution to the covering maps. But now see that it (Strebel) parametrises the (arithmetic) points on $\mathcal{M}_{0,n}$.

- As $N \to \infty$, the sum goes over to an integral over moduli space $\mathcal{M}_{0,n}$.

- Realises the program of associating Feynman diagrams to points in moduli space (via Strebel).

- Integrand on moduli space $\propto e^{-N^2 S_{cl}[\Gamma]}$. With $S_{cl}[\Gamma] \propto \int d^2 z \vert S[\Gamma] \vert$. (From Liouville action).

- $S[\Gamma] = \frac{\Gamma'''}{\Gamma'} - \frac{3}{2} \left( \frac{\Gamma''}{\Gamma'} \right)^2$ - the Schwarzian of the covering map - also equals $\phi_S(z)$ at large $N$!
B. FROM STRINGS TO FIELDS
TENSIONLESS STRINGS ON \( AdS_3 \)

- Novel features of the worldsheet theory in the tensionless limit:

1. **Free field** (GLSM) description \([\mathfrak{psu}(1,1 \mid 2)_1]\) despite being highly curved \( AdS_3 \)-like.

2. In terms of **holomorphic twistor variables**: 2 symplectic bosons (\( \xi^\pm \)) and 2 fermions (+ conjugates).

3. **Spectrally flowed sectors** \( \{ w_i \} \) of the WZW model ↔ twisted sectors \( \{ w_i \} \) of dual orbifold CFT.

4. Worldsheet correlators of these sectors are delta function localised to **discrete points on** \( \mathcal{M}_{g,n} \).

5. **Semiclassical worldsheet** which is essentially at the **boundary of** \( AdS_3 \) - gives Lunin-Mathur correlators.
The entire (not just BPS) spectrum of the perturbative string theory exactly matches with the (single cycle) states of the large N 2d orbifold CFT. [Eberhardt, Gaberdiel, R.G. -'18]

\[ \mathcal{V}_h^w(x; z) \leftrightarrow \mathcal{O}_h^{(w)}(x) \]

- \( h \) = spacetime conformal dimension = \( AdS_3 \) energy. \( w \) = twisted sector cycle = spectral flow.
- The bulk theory with \( k = 1 \) has fewer states than for \( k > 1 \). No continuum of long strings.
- Only \( j = 1/2 \) multiplets under \( \mathfrak{s}l_2(\mathbb{R}) \) due to truncation in \( \mathfrak{psu}(1,1 \mid 2)_1 \) WZW model.
- Only four transverse oscillators (\( T^4 \)); Quasi-topological on \( AdS_3 \times S^3 \).
FROM STRINGS TO FIELDS: CORRELATORS

\[ \left\langle \sigma^{(w_1)}(x_1)\sigma^{(w_2)}(x_2)\ldots\sigma^{(w_n)}(x_n) \right\rangle_{S^2_{g=0}} = \int_{\mathcal{M}_{0,n}} \left\langle \mathcal{V}_0^{w_1}(x_1; z_1)\mathcal{V}_0^{w_2}(x_2; z_2)\ldots\mathcal{V}_0^{w_n}(x_n; z_n) \right\rangle_{\Sigma_{0,n}} \]

- Restrict to ground states in w-twisted sectors; and genus zero - for simplicity. RHS nontrivially agrees with LHS. (See generalisation to BPS and other states - Gaberdiel-Nairz ’22)

- Because of unusual localisation of worldsheet correlators on moduli space - to holomorphic maps \( x = \Gamma(z) \) with branching \( w_i \) at insertions \( z_i \): \( x \sim x_i + a_i^\Gamma(z - z_i)^{w_i} \); \( i = 1,2\ldots,n \).

\[ \text{[Cf. Eberhardt @StringMath21]} \]

- **CLAIM:** Worldsheet correlator on RHS \( \propto \prod_{i=4}^{n} \delta^{(2)}(x_i - \Gamma(z_i)) \) - discrete set of points allowing covers.

\[ \text{[Eberhardt-Gaberdiel-R.G. -'19]} \]

- Exact semiclassical worldsheet sigma model action gives weight \( \propto e^{-S_L[\Phi=\ln|\partial\Gamma|^2]} \), \( \Phi = \) radial coord.
Thus worldsheet is a covering space of the boundary $S^2$ exactly as in Lunin-Mathur.

This localisation is transparent in a free field realisation of $\mathfrak{psu}(1,1 \mid 2)$ - twistor variables $Z^I = (\xi^\alpha, \psi^\alpha); Y_I = (\epsilon_{\alpha\beta} \eta^\beta, \epsilon_{\alpha\beta} \chi^\beta)$. [Dei, Gaberdiel R.G., Knighton- '20]

Twistor incidence relation:
\[ \langle (\xi^{-}(z) + \Gamma(z)\xi^{+}(z)) \rangle_{\text{phys}} = 0 \]

Implies that correlators are
\[ \propto \sum_{\Gamma} \hat{W}_\Gamma \prod_{i=1}^{n} |a_i^\Gamma|^{-2h_i} \prod_{i=4}^{n} \delta^{(2)}(x_i - \Gamma(z_i)) \]
In large twist limit

Matrix model analysis

Covering map contribution to twist correlators

Spectral curve with cuts

Schwarzian of the covering map

Strebel graph

Feynman graph

Strebel differential

Dual graph

Worldsheet analysis

"Integer point" on moduli space

Worldsheet contribution localised on these points

\[ S[\Gamma]dz^2 \rightarrow \phi_S(z)dz^2 \]

\[ \int_{a_i}^{a_j} \sqrt{\phi_S(z)}dz \propto n_{ij} \]

\[ y^2(z)dz^2 \rightarrow \phi_S(z)dz^2 \]
LOOKING AHEAD
TENSIONLESS STRINGS ON $AdS_5$

Twistorial Gauged Linear Sigma Model for $AdS_3 \times S^3$: $Y_I = (\eta_\alpha, \chi_\beta)$; $Z^I = (\xi^\alpha, \psi^\beta)$.

Twistorial Gauged Linear Sigma Model for $AdS_5 \times S^5$: $Y_I = (\mu^+_a, \lambda^+_a, \psi^+_a)$; $Z^I = (\lambda^a, \mu^a, \psi^a)$.

[Gaberdiel-R. G. ’21]

Ambitwistor Open String Theory ($Y_I, Z^I$)

[BMN & Integrable Spin Chains
[Berenstein-Maldacena-Nastase ’02,...]

[Berkovits’04; Mason-Skinner’13,...]
FREE FIELDS ON THE WORLDSHEET

- Twistor fields $Z^I = (\lambda^\alpha, \mu^\dot{\beta}, \psi^a)$; $Y_J = (\lambda^\dagger_\alpha, \mu^\dagger_\beta, \psi^\dagger_b)$ give a free field representation of $\mathfrak{psu}(2,2|4)$, through bilinears $Y_I Z^J$ (with $C \equiv Y^I Z_I = 0$ - projects out the $\mathfrak{u}(1)$).

- For each $w \in \mathbb{Z}_+$ consider Fock space built on $|0\rangle_w$ by a finite number of “wedge modes”:

  $$(\mu^\dagger_\alpha)_r, \ (\mu^\dot{\beta})_r, \ (\psi^\dagger_a)_r \ (a = 1, 2), \ \psi^b_r \ (b = 3, 4), \ \text{with} \ -\frac{w-1}{2} \leq r \leq \frac{w-1}{2}.$$

- $|0\rangle_w$ is a “spectrally flowed” vacuum state.

- $w = 0 \leftrightarrow$ NS sector, $w = 1 \leftrightarrow$ Ramond sector: only zero modes. These generate the singleton representation of the 4d superconformal algebra $\mathfrak{psu}(2,2|4)$.
**PHYSICAL GAUGE AND SPECTRUM**

- **PROPOSAL:** In a “physical gauge” can gauge away out-of-the-wedge modes leaving only the wedge modes \( Z^I_r, (Y_J)_r \) (with \( \frac{-(w-1)}{2} \leq r \leq \frac{(w-1)}{2} \)) - left with \( w \) bits. Underlying worldsheet \( \mathcal{N} = 4 \)?
  
  [Gaberdiel-R.G. ’21]

- Further, impose residual physical state conditions

  \[
  \begin{align*}
  \text{A)} \quad & (L_0 + pw) |phys\rangle_w = 0 \quad (p \in \mathbb{Z}) \\
  \text{B)} \quad & \mathcal{C}_r |phys\rangle_w = 0 \quad (r = 0, 1, \ldots (w - 1))
  \end{align*}
  \]

  \[ \mathcal{C} = Z^I Y_I \]

- Condition A \( \leftrightarrow \) discrete translation invariance on a worldsheet with \( w \) bits (cyclicity).

- Condition B \( \leftrightarrow \) Each of the individual bits at each “site” form a singleton.

  \[
  \hat{\Phi}_j = \frac{1}{\sqrt{w}} \sum_{r=-\frac{w-1}{2}}^{\frac{w-1}{2}} \Phi_r e^{i2\pi j r/w} \quad \text{Any Twistor Field}
  \]
FREE $\mathcal{N} = 4$ SYM FROM THE WORLDSHEET

- Reproduces precisely the large N gauge invariant spectrum of single trace operators ($w$ letters) in $\mathcal{N} = 4$ SYM:
  \[ \sum_{w} (\text{singleton})^w (\text{cyclicity}). \]
  \[ [w = 0 \leftrightarrow 1 \text{ (identity operator in SYM)}] \]
  [Bianchi, Morales, Samtleben; Alday, David, Gava, Narain]

- Singleton ($w = 1$) $\leftrightarrow$ single “letters” of SYM $\{ \partial^s \phi^i, \partial^s \Psi^a, \partial^s \bar{\Psi}^\alpha, \partial^s \mathcal{F}_\alpha, \partial^s \bar{\mathcal{F}}^\beta \}.$

- Gives an organisation of the free SYM spectrum in terms of $w$ bits - same building blocks of the integrable spin chains that govern the dynamics of perturbative SYM [Minahan-Zarembo; Beisert, Staudacher et.al].

- Need worldsheet quantisation that reproduces these physical modes and residual constraints.
  [Gaberdiel, R. G. (In progress)]

- Expect a localisation of correlators to holomorphic maps into ambitwistor space. $S[\Gamma] \propto \phi_s \Rightarrow e^{-2\pi A_s} \propto \left( \frac{1}{x^2_{ij}} \right)^w$

— Feynman Propagators! [Bhat, Maity, R. G., Radhakrishnan '22]
OUTLOOK

- Exhibited a test case (tensionless $\text{AdS}_3/CFT_2$) where the general program of reassembling large N QFTs into string theories can be explicitly carried out.

- Could also close the circle from strings to fields due to a tractable worldsheet theory.

- Extension to perturbative $\mathcal{N} = 4$ SYM in terms of twistor description of $\text{AdS}_5$. Compelling ingredients for worldsheet theory but needs to be put on solid footing.

- Extend “Fields to Strings” program to other large N QFTs. E.g. string duals to large N Matrix models [R.G.-Mazenc ’22, in progress].

- Tensionless string theories on AdS likely to give a new, unusual family of topological string theories. Interesting connections to Mathematics.
THANKYOU