Families of solutions of the heterotic G_2 system

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Introduction and motivation

General overview

Heterotic string theory compactified on a 7-dimensional manifold with AdS_3 spacetime and minimal supersymmetry.

- These compactifications remain mysterious.
  - Possibility of AdS_3 went unnoticed for years.
- Explicit backgrounds are extremely scarce in the literature.
  - [Lotay, Sá Earp 21].
- Rich mathematical structure.
  - G_2 geometry, G_2 instantons.
  - Deformation theory, moduli spaces.
- Some heterotic version of holography?

Heterotic supergravity

Massless spectrum:

$$\mathcal{N} = 1 \text{ SUGRA} \oplus \mathcal{N} = 1 \text{ SYM},$$

which includes

- Bosons: metric, dilaton, B-field, gauge field A.
- Fermions: gravitino, dilatino, gaugino.

The 3-form flux must satisfy an anomaly cancellation condition

$$H = \mathrm{d}B + \frac{\alpha'}{4}\left(\mathcal{CS}(A) - \mathcal{CS}(\Theta)\right) .$$

The heterotic G_2 system

The ansatz

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[Gauntlett, Kim, Martelli, Waldram 01], [Friedrich, Ivanov 02], [Friedrich, Ivanov 03], [Gauntlett, Martelli, Parkis, Waldram 04], [Gauntlett, Martelli, Waldram 04], [Ivanov, Ivanov 05], [Kunitomo, Ohta 09], [Ivanov 10]...

Consider a heterotic compactification with a warped product

$$\mathcal{M}_3 \times Y_7\,,$$

where

- $\mathcal{M}_3$ maximally symmetric, 3-dimensional spacetime.
- Y_7 a 7-dimensional compact Riemannian manifold.
- $\mathcal{N} = 1$ supersymmetry is preserved in 3 dimensions.

Translate statements in physics to mathematics.

Killing spinor equations

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Supersymmetry is preserved when the Killing spinor equations are satisfied

$$\delta\lambda = \nabla_\mu \epsilon = (\nabla^{LC}_\mu + \frac{1}{8} H_{\mu\nu\rho}\Gamma^{\nu\rho})\epsilon = 0\,,$$
$$\delta\psi = (\Gamma^\mu \partial_\mu \phi - \frac{1}{12} H_{\mu\nu\rho}\Gamma^{\mu\nu\rho})\epsilon = 0\,,$$
$$\delta\xi = F^A_{\mu\nu}\Gamma^{\mu\nu}\epsilon = 0\,.$$

The gravitino and dilatino equations imply

Y_7 has an integrable G_2-structure.

G_2-structures

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[Bryant 03]

A G_2-structure on Y_7 is a reduction of the structure group of the frame bundle of Y_7 to G_2.

This is equivalent to the existence of a non-degenerate positive 3-form on Y_7, the associative 3-form.

Locally

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} .$$

where $\{e^1, \dots, e^7\}$ form an orthonormal basis of one-forms on Y_7 and we are writing $e^{ij} = e^i \wedge e^j$.

Consequences of G_2-structures

- From φ we obtain a metric g, a Hodge star $*$ and the coassociative 4-form

$$\psi = *\varphi = e^{1357} + e^{2345} + e^{2367} + e^{4567} - e^{1247} - e^{1256} - e^{1346} \,.$$

- All tensors on Y_7 can be decomposed in G_2 representations, e.g.

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14} \,,$$

with

$$\Lambda^2_{14} = \{\beta \in \Lambda^2 : \beta \lrcorner \varphi = 0\} \,.$$

G_2-structures and G_2-holonomy

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A manifold Y_7 has G_2-holonomy if it has a G_2-structure such that

$$\mathrm{d}\varphi = 0\,, \qquad \mathrm{d}\psi = 0\,.$$

For a general G_2-structure, decompose in G_2 representations:

$$\mathrm{d}\varphi = \tau_0\,\psi + 3\,\tau_1 \wedge \varphi + *\tau_3\,,$$
$$\mathrm{d}\psi = 4\,\tau_1 \wedge \psi + *\tau_2\,,$$

Here τ_k are k-forms called torsion classes.

They satisfy $\tau_3 \in \Lambda^3_{27}$ and $\tau_2 \in \Lambda^2_{14}$.

A G_2-structure is integrable if $\tau_2 = 0$.

$$d\varphi = \tau_0\, \psi + 3\tau_1 \wedge \varphi + *\tau_3 \,,$$
$$d\psi = 4\tau_1 \wedge \psi \,,$$

In this case there exists a distinguished connection with totally antisymmetric torsion which is encoded by

$$T(\varphi) = \frac{1}{6}\tau_0\, \varphi - \tau_1 \lrcorner\, \psi - \tau_3 \,.$$

We call this the torsion of the G_2-structure.

Implications for physics

Physical quantities are encoded by the torsion classes of the G_2-structure.

- The cosmological constant Λ of the 3-dimensional spacetime

$$\Lambda \sim -\tau_0^2 \,.$$

- The dilaton ϕ

$$\tau_1 = \frac{1}{2} \mathrm{d}\phi \,.$$

- The 3-form flux H is given by the torsion tensor

$$H = T(\varphi) = \frac{1}{6}\,\tau_0\,\varphi - \tau_1 \lrcorner \psi - \tau_3 \,.$$

Gauge theory

The gaugino equation constrains the background gauge field A.

The gauge field A:

- Connection on a vector bundle $V \longrightarrow Y_7$.
- Must be a G_2-instanton.

$$F_A \in \Lambda^2_{14}(\mathrm{End}(V)) \iff F_A \wedge \psi = 0\,.$$

The equations of motion constrain an additional field Θ.

The gauge field Θ:

- Connection on the tangent bundle $TY_7 \longrightarrow Y_7$.
- Must be a G_2-instanton.

$$R_\Theta \in \Lambda^2_{14}(\mathrm{End}(TY_7)) \iff R_\Theta \wedge \psi = 0\,.$$

Anomaly cancellation

Finally, we must impose the anomaly cancellation condition

$$H = \mathrm{d}B + \frac{\alpha'}{4}\left(\mathcal{CS}(A) - \mathcal{CS}(\Theta)\right) .$$

To do so, it is enough to solve the heterotic Bianchi identity

$$\mathrm{d}H = \frac{\alpha'}{4}\left(\operatorname{tr} F_A \wedge F_A - \operatorname{tr} R_\Theta \wedge R_\Theta\right) .$$

- $\alpha' > 0$ is the string parameter.
- Recall the flux is independently determined by $H = T(\varphi)$.

Summary

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$\mathcal{N} = 1$ vacuum solution of the heterotic G_2 system.

- (Y_7, φ) compact manifold with integrable G_2-structure.
- (V, A) vector bundle on Y_7, G_2-instanton connection A.
- (TY_7, Θ) tangent bundle, G_2-instanton connection Θ.
- $H = T(\varphi)$ flux satisfying

$$\mathrm{d}H = \frac{\alpha'}{4}(\operatorname{tr} F_A \wedge F_A - \operatorname{tr} R_\Theta \wedge R_\Theta).$$

Solution is given by a quadruple: $[(Y_7, \varphi), (V, A), (TY_7, \Theta), H]$.

Squashed 3-Sasakian manifolds

3-Sasakian manifolds

A Riemannian manifold (Y_7, g) is 3-Sasakian if its metric cone $(C(Y_7), \bar{g}) = (\mathbb{R}_+ \times Y_7, \mathrm{d}r^2 + r^2 g)$ is a hyperkähler manifold.

Equivalently, a 3-Sasakian manifold is the total space of an SU(2)-bundle over a 4-dimensional orbifold.

Example:

$$\mathbb{S}^3 \longrightarrow \mathbb{S}^7 \longrightarrow \mathbb{S}^4 .$$

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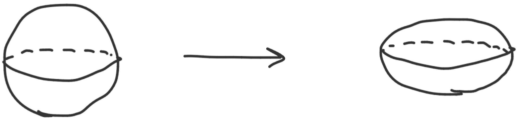
3-Sasakian manifolds can be deformed by squashing.

- Rescale the metric along the SU(2) fibres by an s^2 factor with $s > 0$.

We call s the squashing parameter.



Structure equations and G_2-structure

Find a triple of orthonormal KVFs (η_1, η_2, η_3) satisfying

$$[\eta_i, \eta_j] = 2\, s\, \epsilon_{ij}{}^{k} \eta_k \quad \text{for } i = 1, 2, 3\,.$$

Complete the dual forms to an orthonormal coframe and define

$$\omega^1 = \eta^4 \wedge \eta^5 + \eta^6 \wedge \eta^7\,, \quad \omega^2 = \eta^4 \wedge \eta^6 - \eta^5 \wedge \eta^7\,, \quad \omega^3 = -\eta^4 \wedge \eta^7 - \eta^5 \wedge \eta^6\,.$$

These satisfy the structure equations

$$\mathrm{d}\eta^i = 2\, s\, \omega^i - \frac{1}{s} \epsilon^i{}_{jk}\, \eta^j \wedge \eta^k\,, \qquad \mathrm{d}\omega^i = -\frac{2}{s}\, \epsilon^i{}_{jk}\, \eta^j \wedge \omega^k\,,$$

and we define a one-parameter family of G_2-structures

$$\varphi_s = \eta^{123} + \eta^1 \wedge \omega^1 + \eta^2 \wedge \omega^2 + \eta^3 \wedge \omega^3\,, \qquad \psi_s = *_s \varphi_s\,.$$

Torsion classes

We use the structure equations to compute the torsion classes

$$
\begin{aligned}
\mathrm{d}\varphi_s &= \frac{12}{7}\left(2\,s + \frac{1}{s}\right)\psi_s + \left(10\,s - \frac{2}{s}\right)\left(\eta^{4567} - \frac{1}{7}\psi_s\right),\\
\mathrm{d}\psi_s &= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\tau_0(\varphi_s) &= \frac{12}{7}\left(2\,s + \frac{1}{s}\right),\\
\tau_1(\varphi_s) &= 0\,,\\
\tau_2(\varphi_s) &= 0\,,\\
\tau_3(\varphi_s) &= \left(10\,s - \frac{2}{s}\right)\left(\eta^{123} - \frac{1}{7}\varphi_s\right).
\end{aligned}
$$

Implications for physics

- 3-Sasakian manifolds have a one-parameter family of integrable G_2-structures,

$$\tau_2(\varphi_s) = 0\,.$$

- All of them give rise to AdS_3 spacetimes as

$$\tau_0(\varphi_s) \neq 0\,.$$

- These solutions have constant dilaton since

$$\tau_1(\varphi_s) = 0\,.$$

- The 3-form flux is determined

$$H = T(\varphi_s) = 2\,s\,\varphi_s + \left(\frac{2}{s} - 10\,s\right)\eta^{123}\,.$$

Homogeneous 3-Sasakian manifolds

Consider 3-Sasakian manifolds with a coset description.

- 7-sphere, $\mathbb{S}^7 = \mathrm{Spin}(5)/\mathrm{SU}(2)$.
- Aloff-Wallach space, $N_{1,1} = \mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$.

These are reductive coset manifolds G/K and explicit structure equations can be found.

Progress so far

$$[(Y_7,\varphi),(V,A),(TY_7,\Theta),H]\,.$$

$$\mathrm{d}H=\frac{\alpha'}{4}(\operatorname{tr}F_A\wedge F_A-\operatorname{tr}R_\Theta\wedge R_\Theta).$$

G_2-instantons

G_2-instantons

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We use three different instantons in our solutions:

- Canonical connection.
  - Defined on the bundle $G \longrightarrow G/K$.
- Clarke–Oliveira connection.
  - Defined on pullback of the 3-Sasakian SU(2)-bundle.
- Tangent bundle instantons.
  - One-parameter family of instantons on the tangent bundle.

Canonical connection

[Kobayashi–Nomizu 63]

In short: the coset structure naturally gives rise to an instanton on a principal K-bundle over G/K.

- Obtained from the Maurer–Cartan one-form of G.
- In our case it is a G_2-instanton.

Given a K-representation, we construct the associated vector bundle and find a G_2-instanton on $V \longrightarrow Y_7$.

- In particular, the adjoint representation gives rise to an instanton on the tangent bundle TY_7. [Harland, Nölle 12]

Clarke–Oliveira connection

[Clarke Oliveira 19]

In short: the 3-Sasakian structure naturally gives rise to a G_2-instanton on a principal SU(2)-bundle.

- 3-Sasakian manifolds are SU(2)-orbibundles.
- Consider canonical connection on this bundle.
- Pull it back to an SU(2) bundle over the full (squashed) 3-Sasakian manifold.

Given an SU(2)-representation, we construct the associated vector bundle and find a G_2-instanton on $V \longrightarrow Y_7$.

- In particular, the adjoint representation gives rise to an instanton on the tangent bundle TY_7.

Tangent bundle instantons

There is a one-parameter family of metric connections preserving the G_2-structure, with torsion given by

$$\frac{1}{2}T_{\mu\nu\rho}(a) = \frac{1}{12}\tau_0\,\varphi_{\mu\nu\rho} + a\,\tau_{3\mu\nu\rho} + \frac{1}{4}(1+2\,a)S_\rho{}^\sigma\varphi_{\sigma\mu\nu}\,,$$

where $S_{\mu\nu} = \frac{1}{4}\,\varphi^{\rho\sigma}{}_{(\mu}\,(\tau_3)_{\nu)\rho\sigma}$ and $a \in \mathbb{R}$ is the parameter of the family. [Bryant 03]

- We check all these connections are G_2-instantons.

For each value of $s \neq 1/\sqrt{5}$, we have a

one-parameter family of G_2-instantons.

Progress so far

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$$\left[(Y_7, \varphi), (V, A), (TY_7, \Theta), H\right].$$

$$\mathrm{d}H = \frac{\alpha'}{4}(\operatorname{tr} F_A \wedge F_A - \operatorname{tr} R_\Theta \wedge R_\Theta).$$

New solutions

Finding solutions

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Define for simplicity

$$F_1 = \eta^{123}\,, \qquad F_2 = \eta^1 \wedge \omega^1 + \eta^2 \wedge \omega^2 + \eta^3 \wedge \omega^3\,,$$

- The derivative of the flux is given by

$$\mathrm{d}H = 24\, s^2 *_s F_1 + 8\,(1 - 2\, s^2)\, *_s F_2\,.$$

- Try different combinations of bundles and connections, and compute $\mathrm{tr}\, F \wedge F$: it has two terms $*_s F_1$ and $*_s F_2$.

We impose the heterotic Bianchi identity

$$\mathrm{d}H = \frac{\alpha'}{4}(\mathrm{tr}\, F_A \wedge F_A - \mathrm{tr}\, R_\ominus \wedge R_\ominus).$$

We try to solve the system in terms of s and α'.

Summary of results

(V, A) \ (TY_7, Θ)	Canonical connection	Clarke–Oliveira connection	Tangent bundle instantons
Canonical connection	Solutions only for $s = 1/\sqrt{2}$, fixed α'	Solutions for isolated values of s and α'.	Solutions in different ranges of s with α' determined.
Clarke–Oliveira connection	No solution	No solution	Solutions in different ranges of s with α' determined.
Tangent bundle instantons	No solution	Solutions in different ranges of s with α' determined.	Solutions with arbitrary s and α' within a certain range.

First example

(V, A) \ (TY_7, Θ)	Canonical connection	Clarke–Oliveira connection	Tangent bundle instantons
Canonical connection	Solutions only for $s = 1/\sqrt{2}$, fixed α'	Solutions for isolated values of s and α'.	Solutions in different ranges of s with α' determined.
Clarke–Oliveira connection	No solution	No solution	Solutions in different ranges of s with α' determined.
Tangent bundle instantons	No solution	Solutions in different ranges of s with α' determined.	Solutions with arbitrary s and α' within a certain range.

Canonical in V, family in TY_7

- Canonical connection on the vector bundle V, with adjoint representation.
- One-parameter family on the tangent bundle TY_7, with parameter a.

The terms in the heterotic Bianchi identity can be computed

$$
\begin{aligned}
\mathrm{tr}(F_A \wedge F_A) &= 96 *_s F_1 \,, \\
\mathrm{tr}(R_\Theta \wedge R_\Theta) &= -\,72\, s^2 \left(\kappa(a,s)^2 - \frac{4}{3s^2} \right) *_s F_1 \\
&\quad -\,12\, s\ \kappa(a,s)^2 \left(\kappa(a,s) - \frac{2}{s} \right) *_s F_2 \,,
\end{aligned}
$$

with $\kappa(a,s) = (1 + 10\,a)s + (1 - 2\,a)\frac{1}{s}$.

Canonical in V, family in TY_7

The heterotic Bianchi identity becomes

$$24\, s^2 = \frac{\alpha'}{4}\left[96 - 72\, s^2\left(\kappa(a,s)^2 - \frac{4}{3s^2}\right)\right]\,,$$
$$8(1 - 2\, s^2) = \frac{\alpha'}{4}\left[-12\, s\ \kappa(a,s)^2\left(\kappa(a,s) - \frac{2}{s}\right)\right]\,.$$

The equations have a unique solution as long as $s \neq 1$ and $s \neq 1/\sqrt{5}$

$$a(s) = -\frac{5\, s^2 - 3}{10\, s^2 - 2}, \qquad \alpha'(s) = \frac{s^2}{12(s^2-1)^2}\,.$$

Canonical in V, family in TY_7

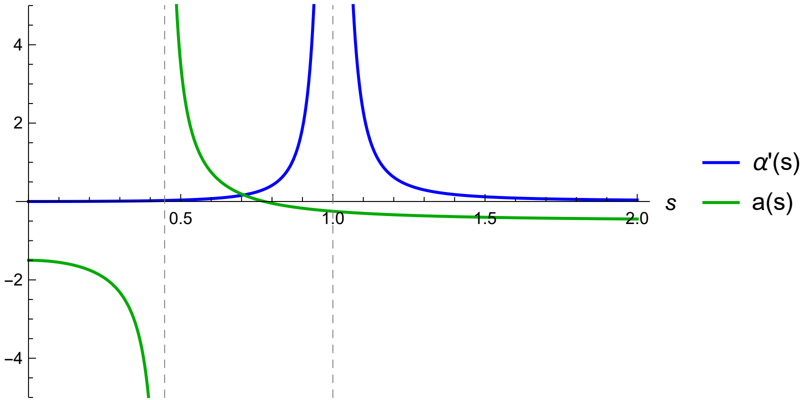

Figure: Values of $\alpha'(s)$ and $a(s)$ in terms of s, adjoint representation.

Second example

(TY_7, Θ) / (V, A)	Canonical connection	Clarke–Oliveira connection	Tangent bundle instantons
Canonical connection	Solutions only for $s = 1/\sqrt{2}$, fixed α'	Solutions for isolated values of s and α'.	Solutions in different ranges of s with α' determined.
Clarke–Oliveira connection	No solution	No solution	Solutions in different ranges of s with α' determined.
Tangent bundle instantons	No solution	Solutions in different ranges of s with α' determined.	Solutions with arbitrary s and α' within a certain range.

Family in V, family in TY_7

- Connection Θ on TY_7 in the one-parameter family with parameter a.
- Connection A on $V = TY_7$ in the one-parameter family with parameter $b \neq a$.

Given a pair (s, α') such that

$$s \neq 1\,, \qquad s \neq \frac{1}{\sqrt{5}}\,, \qquad \alpha' > \frac{s^2}{12(s^2-1)^2}\,,$$

we find two sets of values for $a(s, \alpha')$ and $b(s, \alpha')$ that give a solution of the system.

Family in V, family in TY_7

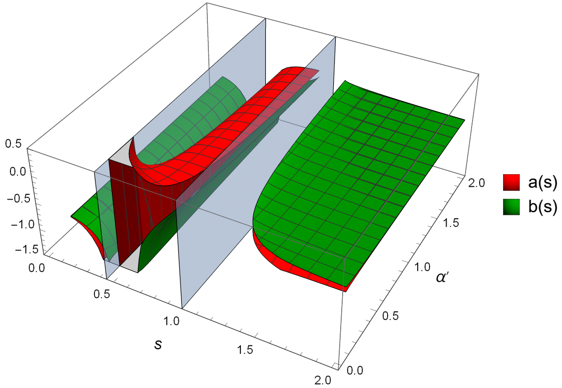

Figure: First set of values of $a(s, \alpha')$ and $b(s, \alpha')$ in terms of s and α'.

Family in V, family in TY_7

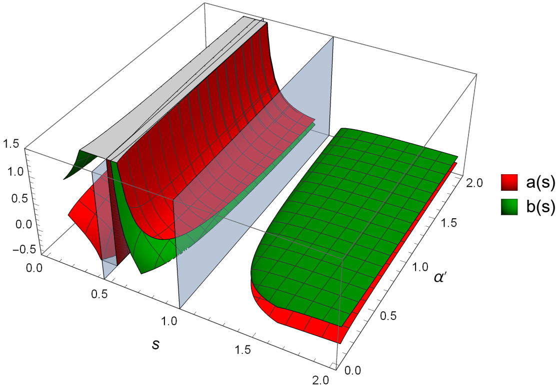

Figure: Second set of values of $a(s,\alpha')$ and $b(s,\alpha')$ in terms of s and α'.

Finite deformations

- Fix $\alpha' > 0$, i.e. take a "slice" in the previous figures.
- Start from a solution (s_0, α').
- We can deform the squashing parameter s keeping α' fixed.
- The instanton connections are deformed as the manifold is squashed, ensuring the Bianchi identity is still satisfied.

These are the first examples of finite deformations of solutions of the heterotic G_2 system.

Finite deformations

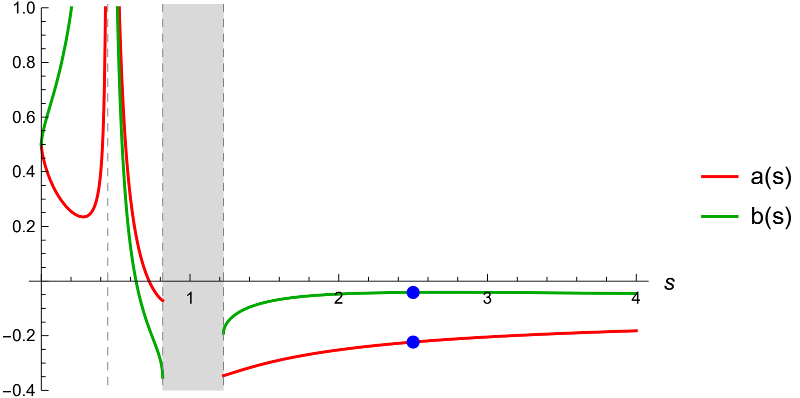

Figure: One set of values of $a(s,\alpha')$ and $b(s,\alpha')$ in terms of s and with $\alpha' = 1/2$ fixed. The blue dots refer to the starting point for the deformation $s_0 = 5/2$.

Conclusion and outlook

Conclusions

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- The heterotic G_2 system is nontrivial.
- We find new solutions on squashed homogeneous 3-Sasakian manifolds.
  - Different families.
  - AdS_3 spacetime.
- First examples of finite deformations of solutions.

Future directions

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- Generalization of the solutions
  - Non-homogeneous squashed 3-Sasakian manifolds.
  - Different G_2-structures or G_2-instantons.
  - Sasaki, Sasaki-Einstein manifolds.
- Study of the moduli space of solutions.
  - Infinitesimal vs. finite deformations.
- Search for T-dual solutions (abelian and non-abelian).
- Analysis of solutions from a worldsheet perspective.
- Possibility of some version of holography.

Thank you!