Families of solutions of the heterotic $G_2$ system

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Based on 2111.13221 with X. de la Ossa

String Math 2022 Parallel Session 13/07/22
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Introduction and motivation
General overview

Heterotic string theory compactified on a 7-dimensional manifold with AdS$_3$ spacetime and minimal supersymmetry.

- These compactifications remain mysterious.
  - Possibility of AdS$_3$ went unnoticed for years.
- Explicit backgrounds are extremely scarce in the literature.
  - [Lotay, Sá Earp 21].
- Rich mathematical structure.
  - G$_2$ geometry, G$_2$ instantons.
  - Deformation theory, moduli spaces.
- Some heterotic version of holography?
Heterotic supergravity

Massless spectrum:

\[ \mathcal{N} = 1 \text{ SUGRA} \oplus \mathcal{N} = 1 \text{ SYM}, \]

which includes

- **Bosons**: metric, dilaton, B-field, gauge field \( A \).
- **Fermions**: gravitino, dilatino, gaugino.

The 3-form flux must satisfy an anomaly cancellation condition

\[ H = dB + \frac{\alpha'}{4} (CS(A) - CS(\Theta)) . \]
The heterotic $G_2$ system
Consider a heterotic compactification with a warped product

\[ M_3 \times Y_7, \]

where

- \( M_3 \) maximally symmetric, 3-dimensional spacetime.
- \( Y_7 \) a 7-dimensional compact Riemannian manifold.
- \( \mathcal{N} = 1 \) supersymmetry is preserved in 3 dimensions.

Translate statements in physics to mathematics.
Supersymmetry is preserved when the Killing spinor equations are satisfied

\[ \delta \lambda = \nabla_\mu \epsilon = (\nabla^L_\mu + \frac{1}{8} H_{\mu\nu\rho} \Gamma^{\nu\rho}) \epsilon = 0 , \]

\[ \delta \psi = (\Gamma^\mu \partial_\mu \phi - \frac{1}{12} H_{\mu\nu\rho} \Gamma^{\mu\nu\rho}) \epsilon = 0 , \]

\[ \delta \xi = F^A_{\mu\nu} \Gamma^{\mu\nu} \epsilon = 0 . \]

The gravitino and dilatino equations imply

\[ Y_7 \text{ has an integrable } G_2\text{-structure.} \]
A $G_2$-structure on $Y_7$ is a reduction of the structure group of the frame bundle of $Y_7$ to $G_2$.

This is equivalent to the existence of a non-degenerate positive 3-form on $Y_7$, the associative 3-form.

Locally

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. $$

where \(\{e^1, \ldots, e^7\}\) form an orthonormal basis of one-forms on $Y_7$ and we are writing $e^{ij} = e^i \wedge e^j$. 
Consequences of $G_2$-structures

- From $\varphi$ we obtain a metric $g$, a Hodge star $\ast$ and the coassociative 4-form

$$\psi = \ast \varphi = e^{1357} + e^{2345} + e^{2367} + e^{4567} - e^{1247} - e^{1256} - e^{1346}.$$ 

- All tensors on $Y_7$ can be decomposed in $G_2$ representations, e.g.

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14},$$

with

$$\Lambda^2_{14} = \{ \beta \in \Lambda^2 : \beta \perp \varphi = 0 \}.$$
A manifold $Y_7$ has $G_2$-holonomy if it has a $G_2$-structure such that

$$d\varphi = 0, \quad d\psi = 0.$$

For a general $G_2$-structure, decompose in $G_2$ representations:

$$d\varphi = \tau_0 \psi + 3 \tau_1 \wedge \varphi + *\tau_3,$$
$$d\psi = 4 \tau_1 \wedge \psi + *\tau_2,$$

Here $\tau_k$ are $k$-forms called torsion classes.

They satisfy $\tau_3 \in \Lambda^3_{27}$ and $\tau_2 \in \Lambda^2_{14}$. 
A $G_2$-structure is integrable if $\tau_2 = 0$.

\[
\begin{align*}
\text{d}\varphi &= \tau_0 \psi + 3\tau_1 \wedge \varphi + *\tau_3, \\
\text{d}\psi &= 4\tau_1 \wedge \psi,
\end{align*}
\]

In this case there exists a distinguished connection with totally antisymmetric torsion which is encoded by

\[
T(\varphi) = \frac{1}{6} \tau_0 \varphi - \tau_1 \wedge \psi - \tau_3.
\]

We call this the torsion of the $G_2$-structure.
Implications for physics

Physical quantities are encoded by the torsion classes of the \( G_2 \)-structure.

- The cosmological constant \( \Lambda \) of the 3-dimensional spacetime
  \[ \Lambda \sim -\tau_0^2. \]

- The dilaton \( \phi \)
  \[ \tau_1 = \frac{1}{2} \mathrm{d}\phi. \]

- The 3-form flux \( H \) is given by the torsion tensor
  \[ H = T(\varphi) = \frac{1}{6} \tau_0 \varphi - \tau_1 \eta \psi - \tau_3. \]
Gauge theory

The gaugino equation constrains the background gauge field $A$.

The gauge field $A$:
- Connection on a vector bundle $V \to Y_7$.
- Must be a $G_2$-instanton.

$$F_A \in \Lambda^2_{14} (\text{End}(V)) \iff F_A \wedge \psi = 0.$$ 

The equations of motion constrain an additional field $\Theta$.

The gauge field $\Theta$:
- Connection on the tangent bundle $TY_7 \to Y_7$.
- Must be a $G_2$-instanton.

$$R_\Theta \in \Lambda^2_{14} (\text{End}(TY_7)) \iff R_\Theta \wedge \psi = 0.$$
Finally, we must impose the anomaly cancellation condition

$$H = dB + \frac{\alpha'}{4} (CS(A) - CS(\Theta)).$$

To do so, it is enough to solve the heterotic Bianchi identity

$$dH = \frac{\alpha'}{4} (\text{tr} \, F_A \wedge F_A - \text{tr} \, R_{\Theta} \wedge R_{\Theta}).$$

- $\alpha' > 0$ is the string parameter.
- Recall the flux is independently determined by $H = T(\varphi)$. 
\( \mathcal{N} = 1 \) vacuum solution of the **heterotic G\(_2\) system.**

- \((Y_7, \varphi)\) compact manifold with integrable G\(_2\)-structure.
- \((V, A)\) vector bundle on \(Y_7\), G\(_2\)-instanton connection \(A\).
- \((TY_7, \Theta)\) tangent bundle, G\(_2\)-instanton connection \(\Theta\).
- \(H = T(\varphi)\) flux satisfying
  \[
  dH = \frac{\alpha'}{4}(\text{tr } F_A \wedge F_A - \text{tr } R_\Theta \wedge R_\Theta).
  \]

Solution is given by a quadruple: \([ (Y_7, \varphi), (V, A), (TY_7, \Theta), H ] \).
Squashed 3-Sasakian manifolds
A Riemannian manifold \((Y_7, g)\) is 3-Sasakian if its metric cone \((C(Y_7), \bar{g}) = (\mathbb{R}_+ \times Y_7, dr^2 + r^2 g)\) is a hyperkähler manifold.

Equivalently, a 3-Sasakian manifold is the total space of an \(SU(2)\)-bundle over a 4-dimensional orbifold.

Example:

\[
\mathbb{S}^3 \longrightarrow \mathbb{S}^7 \longrightarrow \mathbb{S}^4.
\]
Squashing

3-Sasakian manifolds can be deformed by squashing.

- Rescale the metric along the SU(2) fibres by an $s^2$ factor with $s > 0$.

We call $s$ the squashing parameter.
Structure equations and $G_2$-structure

Find a triple of orthonormal KVFs $(\eta_1, \eta_2, \eta_3)$ satisfying

$$[\eta_i, \eta_j] = 2s \epsilon_{ij}^k \eta_k \quad \text{for } i = 1, 2, 3.$$

Complete the dual forms to an orthonormal coframe and define

$$\omega^1 = \eta^4 \wedge \eta^5 + \eta^6 \wedge \eta^7, \quad \omega^2 = \eta^4 \wedge \eta^6 - \eta^5 \wedge \eta^7, \quad \omega^3 = -\eta^4 \wedge \eta^7 - \eta^5 \wedge \eta^6.$$

These satisfy the structure equations

$$d\eta^i = 2s \omega^i - \frac{1}{s} \epsilon_{jkk}^i \eta^j \wedge \eta^k, \quad d\omega^i = -\frac{2}{s} \epsilon_{jkk}^i \eta^j \wedge \omega^k,$$

and we define a one-parameter family of $G_2$-structures

$$\varphi_s = \eta^{123} + \eta^1 \wedge \omega^1 + \eta^2 \wedge \omega^2 + \eta^3 \wedge \omega^3, \quad \psi_s = *_s \varphi_s.$$
Torsion classes

We use the structure equations to compute the torsion classes

\[ d\varphi_s = \frac{12}{7} \left( 2s + \frac{1}{s} \right) \psi_s + \left( 10s - \frac{2}{s} \right) \left( \eta^{4567} - \frac{1}{7} \psi_s \right), \]
\[ d\psi_s = 0, \]

\[ \tau_0(\varphi_s) = \frac{12}{7} \left( 2s + \frac{1}{s} \right), \]
\[ \tau_1(\varphi_s) = 0, \]
\[ \tau_2(\varphi_s) = 0, \]
\[ \tau_3(\varphi_s) = \left( 10s - \frac{2}{s} \right) \left( \eta^{123} - \frac{1}{7} \varphi_s \right). \]
3-Sasakian manifolds have a one-parameter family of integrable $G_2$-structures,
$$\tau_2(\varphi_s) = 0.$$  

All of them give rise to $AdS_3$ spacetimes as
$$\tau_0(\varphi_s) \neq 0.$$  

These solutions have constant dilaton since
$$\tau_1(\varphi_s) = 0.$$  

The 3-form flux is determined
$$H = T(\varphi_s) = 2s \varphi_s + \left(\frac{2}{s} - 10s\right) \eta^{123}.$$
Homogeneous 3-Sasakian manifolds

Consider 3-Sasakian manifolds with a coset description.

- 7-sphere, $S^7 = \text{Spin}(5)/\text{SU}(2)$.

- Aloff-Wallach space, $N_{1,1} = \text{SU}(3)/\text{U}(1)_{1,1}$.

These are reductive coset manifolds $G/K$ and explicit structure equations can be found.
Progress so far

\[
((Y_7, \varphi), (V, A), (TY_7, \Theta), H).
\]

\[
dH = \frac{\alpha'}{4} \left( \text{tr} F_A \wedge F_A - \text{tr} R_\Theta \wedge R_\Theta \right).
\]
$G_2$-instantons
G$_2$-instantons

We use three different instantons in our solutions:

- Canonical connection.
  - Defined on the bundle $G \rightarrow G/K$.

- Clarke–Oliveira connection.
  - Defined on pullback of the 3-Sasakian SU(2)-bundle.

- Tangent bundle instantons.
  - One-parameter family of instantons on the tangent bundle.
Canonical connection

In short: the coset structure naturally gives rise to an instanton on a principal $K$-bundle over $G/K$.

- Obtained from the Maurer–Cartan one-form of $G$.
- In our case it is a $G_2$-instanton.

Given a $K$-representation, we construct the associated vector bundle and find a $G_2$-instanton on $V \longrightarrow Y_7$.

- In particular, the adjoint representation gives rise to an instanton on the tangent bundle $TY_7$.  [Harland, Nölle 12]
Clarke–Oliveira connection

In short: the 3-Sasakian structure naturally gives rise to a $G_2$-instanton on a principal SU(2)-bundle.

- 3-Sasakian manifolds are SU(2)-orbibundles.
- Consider canonical connection on this bundle.
- Pull it back to an SU(2) bundle over the full (squashed) 3-Sasakian manifold.

Given an SU(2)-representation, we construct the associated vector bundle and find a $G_2$-instanton on $V \rightarrow Y_7$.

- In particular, the adjoint representation gives rise to an instanton on the tangent bundle $TY_7$. 
Tangent bundle instantons

There is a one-parameter family of metric connections preserving the $G_2$-structure, with torsion given by

$$\frac{1}{2} T_{\mu\nu\rho}(a) = \frac{1}{12} \tau_0 \varphi_{\mu\nu\rho} + a \tau_3_{\mu\nu\rho} + \frac{1}{4} (1 + 2 a) S^\sigma_{\rho} \varphi_{\sigma\mu\nu} ,$$

where $S_{\mu\nu} = \frac{1}{4} \varphi^{\rho\sigma} (\mu (\tau_3)_\nu)_{\rho\sigma}$ and $a \in \mathbb{R}$ is the parameter of the family. [Bryant 03]

▶ We check all these connections are $G_2$-instantons.

For each value of $s \neq 1/\sqrt{5}$, we have a one-parameter family of $G_2$-instantons.
Progress so far

\[ [(Y_7, \varphi), (V, A), (TY_7, \Theta), H]. \]

\[ dH = \frac{\alpha'}{4} (\text{tr} \ F_A \wedge F_A - \text{tr} \ R_\Theta \wedge R_\Theta). \]
New solutions
Finding solutions

Define for simplicity

\[ F_1 = \eta^{123}, \quad F_2 = \eta^1 \land \omega^1 + \eta^2 \land \omega^2 + \eta^3 \land \omega^3, \]

- The derivative of the flux is given by
  \[ dH = 24 s^2 \star_s F_1 + 8 (1 - 2 s^2) \star_s F_2. \]

- Try different combinations of bundles and connections, and compute \( \text{tr} F \land F \): it has two terms \( \star_s F_1 \) and \( \star_s F_2 \).

We impose the heterotic Bianchi identity

\[ dH = \frac{\alpha'}{4}(\text{tr} F_A \land F_A - \text{tr} R_\Theta \land R_\Theta). \]

We try to solve the system in terms of \( s \) and \( \alpha' \).
## Summary of results

<table>
<thead>
<tr>
<th>((TY_7, \Theta))</th>
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<th>Clarke–Oliveira connection</th>
<th>Tangent bundle instantons</th>
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<td>Canonical connection</td>
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<td>Solutions for isolated values of (s) and (\alpha').</td>
<td>Solutions in different ranges of (s) with (\alpha') determined.</td>
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### First example

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</table>
Canonical in $V$, family in $TY_7$

- Canonical connection on the vector bundle $V$, with adjoint representation.
- One-parameter family on the tangent bundle $TY_7$, with parameter $a$.

The terms in the heterotic Bianchi identity can be computed

$$\text{tr}(F_A \wedge F_A) = 96 *_s F_1,$$

$$\text{tr}(R_\Theta \wedge R_\Theta) = -72 s^2 \left( \kappa(a, s)^2 - \frac{4}{3s^2} \right) *_s F_1$$

$$- 12 s \kappa(a, s)^2 \left( \kappa(a, s) - \frac{2}{s} \right) *_s F_2,$$

with $\kappa(a, s) = (1 + 10a)s + (1 - 2a)\frac{1}{s}$.
The heterotic Bianchi identity becomes

\[
24 s^2 = \frac{\alpha'}{4} \left[ 96 - 72 s^2 \left( \kappa(a, s)^2 - \frac{4}{3 s^2} \right) \right],
\]

\[
8(1 - 2 s^2) = \frac{\alpha'}{4} \left[ -12 s \kappa(a, s)^2 \left( \kappa(a, s) - \frac{2}{s} \right) \right].
\]

The equations have a unique solution as long as \( s \neq 1 \) and \( s \neq 1/\sqrt{5} \)

\[
a(s) = -\frac{5 s^2 - 3}{10 s^2 - 2}, \quad \alpha'(s) = \frac{s^2}{12(s^2 - 1)^2}.
\]
Canonical in $V$, family in $TY_7$

Figure: Values of $\alpha'(s)$ and $a(s)$ in terms of $s$, adjoint representation.
## Second example

<table>
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<td>$(V, A)$</td>
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Family in $V$, family in $TY_7$

- Connection $\Theta$ on $TY_7$ in the one-parameter family with parameter $a$.
- Connection $A$ on $V = TY_7$ in the one-parameter family with parameter $b \neq a$.

Given a pair $(s, \alpha')$ such that

$$s \neq 1, \quad s \neq \frac{1}{\sqrt{5}}, \quad \alpha' > \frac{s^2}{12(s^2 - 1)^2},$$

we find two sets of values for $a(s, \alpha')$ and $b(s, \alpha')$ that give a solution of the system.
Family in \( V \), family in \( TY_7 \)

**Figure:** First set of values of \( a(s, \alpha') \) and \( b(s, \alpha') \) in terms of \( s \) and \( \alpha' \).
Family in $V$, family in $TY_7$

**Figure:** Second set of values of $a(s, \alpha')$ and $b(s, \alpha')$ in terms of $s$ and $\alpha'$.
Finite deformations

- Fix $\alpha' > 0$, i.e. take a “slice” in the previous figures.
- Start from a solution $(s_0, \alpha')$.
- We can deform the squashing parameter $s$ keeping $\alpha'$ fixed.
- The instanton connections are deformed as the manifold is squashed, ensuring the Bianchi identity is still satisfied.

These are the first examples of finite deformations of solutions of the heterotic $G_2$ system.
Finite deformations

Figure: One set of values of $a(s, \alpha')$ and $b(s, \alpha')$ in terms of $s$ and with $\alpha' = 1/2$ fixed. The blue dots refer to the starting point for the deformation $s_0 = 5/2$. 
Conclusion and outlook
Conclusions

- The heterotic $G_2$ system is nontrivial.
- We find new solutions on squashed homogeneous 3-Sasakian manifolds.
  - Different families.
  - $\text{AdS}_3$ spacetime.
- First examples of finite deformations of solutions.
Future directions

- **Generalization** of the solutions
  - Non-homogeneous squashed 3-Sasakian manifolds.
  - Different $G_2$-structures or $G_2$-instantons.
  - Sasaki, Sasaki-Einstein manifolds.

- Study of the **moduli space** of solutions.
  - Infinitesimal vs. finite deformations.

- Search for **T-dual** solutions (abelian and non-abelian).

- Analysis of solutions from a **worldsheet perspective**.

- Possibility of some version of **holography**.
Thank you!