Finite Symmetry in Field Theory

Dan Freed
University of Texas at Austin

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Joint work with Greg Moore and Constantin Teleman
Simons Collaboration (https://scgcs.berkeley.edu)

Global Categorical Symmetry
Categorical Symmetry
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Symmetry
Symmetry in QFT

Motivating idea for this talk:

Separate out the abstract structure of symmetry from its concrete manifestations as actions or representations.
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Details appear in lecture notes on the collaboration website and in a forthcoming paper.
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Motivation: algebras

Abstract symmetry data (for algebras) is a pair \((A, R)\):

- \(A\): algebra
- \(R\): right regular module

**Definition:** Let \(V\) be a vector space. An \((A, R)\)-action on \(V\) is a pair \((L, \theta)\) consisting of a left \(A\)-module \(L\) together with an isomorphism of vector spaces

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\theta: R \otimes_A L \xrightarrow{\simeq} V
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**Analogy:** 
- algebra \(\rightsquigarrow\) topological field theory 
- element of algebra \(\rightsquigarrow\) defect in TFT
Example: Let $G$ be a finite group. Its group algebra is

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} \lambda_g g \right\}, \quad \lambda_g \in \mathbb{C}$$

Identify $\mathbb{C}[G] = \text{Fun}(G)$; convolution product is pushforward under

$$\text{mult}: G \times G \rightarrow G$$
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**Higher Example:** $\text{Vect} = \text{category of finite dimensional complex vector spaces}. \text{ Define } \text{ Vect} [G] \text{ as the linear category (Vect-module) of vector bundles over } G \text{ with tensor product pushforward under } \text{mult. It is a fusion category }
Definition: An augmentation of an algebra $A$ is an algebra homomorphism $\epsilon: A \to \mathbb{C}$. Use $\epsilon$ to give a right $A$-module structure to $\mathbb{C}$: $\lambda \cdot a = \lambda \epsilon(a)$, $\lambda \in \mathbb{C}$.
Quotients: augmentations

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Use $\epsilon$ to give a right $A$-module structure to $\mathbb{C}$: $\lambda \cdot a = \lambda \epsilon(a)$, $\lambda \in \mathbb{C}$

**Example:** $A = \mathbb{C}[G]$:

$$\epsilon: \mathbb{C}[G] \rightarrow \mathbb{C}$$

$$\sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \lambda_g$$
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$$Q = \mathbb{C} \otimes_A L = \mathbb{C} \otimes_\epsilon L$$
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**Example:** $A = \mathbb{C}[G]$, $S$ a finite $G$-set, $L = \mathbb{C}\langle S \rangle$: then $Q = \mathbb{C} \otimes_A \mathbb{C}\langle S \rangle \cong \mathbb{C}\langle S/G \rangle$
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**Augmentations for higher algebras:** $\Phi$ tensor category $\quad \epsilon: \Phi \rightarrow \text{Vect}$ fiber functor
Projective symmetries

Quantum theory is *projective*, not *linear*: pure states form a projective space
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**Example:** Projective representation of $G$ is linear representation of $G^\tau$ in a central extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow G^\tau \longrightarrow G \longrightarrow 1$$

Isomorphism class of extension $[\tau] \in H^2(G; \mathbb{C}^\times)$  Module over twisted group algebra:

$$A^\tau = \bigoplus_{g \in G} L^\tau_g$$
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Module over twisted group algebra:

$$A^\tau = \bigoplus_{g \in G} L_g^\tau$$

An augmentation $\epsilon : A^\tau \rightarrow \mathbb{C}$ splits the extension, so does not exist if $[\tau] \neq 0$. 
Field theory

**Analogy:** field theory ~ module over an algebra OR ~ representation of a Lie group

**Warning:** This analogy is quite limited
**Field theory**

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**Segal Axiom System:** A (Wick-rotated) field theory $F$ is a linear representation of a bordism (multi)category $\text{Bord}_n(F)$

- $n$ dimension of spacetime
- $F$ background fields (orientation, Riemannian metric, ...)

\[
\begin{align*}
\gamma_{n-1} & \xrightarrow{F} F(\gamma) \\
\gamma_1 & \xrightarrow{\gamma_2} \text{\`} X^n : \gamma_1 \cup \gamma_2 \cup \gamma_3 \to \phi^{n-1} \\
\gamma_3 & \xrightarrow{\gamma_4} F(X) : F(\gamma_1) \otimes F(\gamma_2) \otimes F(\gamma_3) \to \mathcal{C}
\end{align*}
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Fully local theory for *topological* theories; full locality in principle for general theories
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**Kontsevich-Segal:** Axioms for 2-tier nontopological theory $F : \text{Bord}_{n-1,n}(\mathcal{F}) \to t\text{Vect}$
Domain walls, boundary theories, defects

\( \sigma, \sigma_1, \sigma_2 \) \hspace{0.5cm} (n + 1)-dimensional theories
\( \delta: \sigma_1 \rightarrow \sigma_2 \) \hspace{0.5cm} domain wall
\( \rho: \sigma \rightarrow \mathbb{1} \) \hspace{0.5cm} right boundary theory
\( \hat{F}: \mathbb{1} \rightarrow \sigma \) \hspace{0.5cm} left boundary theory

The "sandwich" \( \delta \ F \) is an (absolute) \( n \)-dimensional theory

More generally, one can put defects on any (stratified) manifold \( D \subset M \).
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\[ \sigma, \sigma_1, \sigma_2 \quad (n + 1) \text{-dimensional theories} \]
\[ \delta: \sigma_1 \to \sigma_2 \quad \text{domain wall} \quad (\sigma_2, \sigma_1) \text{-bimodule} \]
\[ \rho: \sigma \to \mathbb{1} \quad \text{right boundary theory} \quad \text{right } \sigma \text{-module} \]
\[ \tilde{F}: \mathbb{1} \to \sigma \quad \text{left boundary theory} \quad \text{left } \sigma \text{-module} \]
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\( \sigma, \sigma_1, \sigma_2 \) \hspace{1cm} (\( n + 1 \))-dimensional theories

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\( \rho : \sigma \rightarrow \mathbb{1} \) \hspace{1cm} right boundary theory \hspace{1cm} right \( \sigma \)-module

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The “sandwich” \( \rho \otimes_\sigma \tilde{F} \) is an (absolute) \( n \)-dimensional theory
Domain walls, boundary theories, defects

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The “sandwich” $\rho \otimes_{\sigma} \tilde{F}$ is an (absolute) $n$-dimensional theory

More generally, one can put defects on any (stratified) manifold $D \subset M$
Given two field theories $F_1, F_2$ on the same domain $\text{Bord}_n(\mathcal{F})$, there is a composition $F_1 \otimes F_2$. The composition law is sometimes called \textit{stacking}. There is a unit $1$ for the composition law.
Composition laws; invertibility

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- There is also a composition law on parallel defects, for example the OPE on point defects. In a topological theory one obtains a higher algebra of defects.
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So a notion of \textit{invertible} field theory and \textit{invertible} defect
Main definition: abstract symmetry data

Fix a dimension $n$ and background fields $\mathcal{F}$ (which we keep implicit)

**Definition:** Finite field-theoretic symmetry data of dimension $n$ is a pair $(\sigma, \rho)$ in which $\sigma$ is an $(n + 1)$-dimensional topological field theory and $\rho$ is a topological right $\sigma$-module.
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**Example:** Let $G$ be a finite group. Then for a $G$-symmetry we let $\sigma$ be finite gauge theory in dimension $n + 1$. Note this is the *quantum* theory which sums over principal $G$-bundles.
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**Regular $\rho$:** Suppose $\mathcal{C}'$ is a symmetric monoidal $n$-category and $\sigma$ is an $(n + 1)$-dimensional topological field theory with codomain $\mathcal{C} = \text{Alg}(\mathcal{C}')$. Let $A = \sigma(\text{pt})$. Then $A$ is an algebra in $\mathcal{C}'$ which, as an object in $\mathcal{C}$, is $(n + 1)$-dualizable. Assume that the right regular module $A_A$ is $n$-dualizable as a 1-morphism in $\mathcal{C}$. Then the boundary theory $\rho$ determined by $A_A$ is the right regular boundary theory of $\sigma$, or the right regular $\sigma$-module.
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Fix a dimension \( n \) and background fields \( \mathcal{F} \) (which we keep implicit)

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**Regular \( \rho \):** Suppose \( \mathcal{C}' \) is a symmetric monoidal \( n \)-category and \( \sigma \) is an \((n + 1)\)-dimensional topological field theory with codomain \( \mathcal{C} = \text{Alg}(\mathcal{C}') \). Let \( A = \sigma(\text{pt}) \). Then \( A \) is an algebra in \( \mathcal{C}' \) which, as an object in \( \mathcal{C} \), is \((n + 1)\)-dualizable. Assume that the right regular module \( A_A \) is \( n \)-dualizable as a 1-morphism in \( \mathcal{C} \). Then the boundary theory \( \rho \) determined by \( A_A \) is the right regular boundary theory of \( \sigma \), or the right regular \( \sigma \)-module.

A regular boundary theory is also called *Dirichlet*
Main definition: concrete realization of symmetry

Let $\sigma$ be an $(n + 1)$-dimensional topological field theory and let $\rho$ be a right $\sigma$-module.

**Definition:** A $(\sigma, \rho)$-module structure on an $n$-dimensional field theory $F$ is a pair $(\tilde{F}, \theta)$ in which $\tilde{F}$ is a left $\sigma$-module and $\theta$ is an isomorphism

$$\theta: \rho \otimes_\sigma \tilde{F} \xrightarrow{\cong} F$$

of absolute $n$-dimensional theories.
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- The theory \( F \) and so the boundary theory \( \hat{F} \) may be topological or nontopological
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- The sandwich picture of $F$ as $\rho \otimes_\sigma \tilde{F}$ separates out the topological part $(\sigma, \rho)$ of the theory from the potentially nontopological part $\tilde{F}$ of the theory.
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- The sandwich picture of $F$ as $\rho \otimes_\sigma \tilde{F}$ separates out the topological part $(\sigma, \rho)$ of the theory from the potentially nontopological part $\tilde{F}$ of the theory.
- Symmetry persists under renormalization group flow, hence a low energy approximation to $F$ should also be an $(\sigma, \rho)$-module. If $F$ is gapped, then we can bring to bear powerful methods and theorems in topological field theory to investigate topological left $\sigma$-modules. This leads to dynamical predictions.
Example: quantum mechanics with $G$-symmetry

$n = 1$

$\mathcal{F}$ \{orientation, Riemannian metric\} for $F$ and $\tilde{F}$

$\mathcal{H}$ Hilbert space

$H$ Hamiltonian

$G \subset \mathcal{H}$ finite group

$S : G \to \text{Aut}(\mathcal{H})$ action on $\mathcal{H}$

$\sigma(\text{pt})$ $\mathbb{C}[G]$

$F(\text{pt})$ $\mathcal{H}$

$\tilde{F}(\text{pt})$ $\mathbb{C}[G]\mathcal{H}$ (left module)
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$\tilde{F}(\text{pt})$ $\mathbb{C}[G]H$ (left module)

Evaluation of some bordisms:

(a) the left module $\mathbb{C}[G]H$

(b) $e^{-\tau H/h} : \mathbb{C}[G]H \to \mathbb{C}[G]H$

(c) the central function $g \mapsto \text{Tr}_H(S(g)e^{-\tau H/h})$ on $G$
Example: gauge theory with $BA$-symmetry

- $n$: any dimension
- $A$: finite abelian group $A = \mu_2$
- $BA$: a homotopical/shifted $A$ ("1-form $A$-symmetry")
- $H$: Lie group with $A \subset Z(H)$ $H = SU_2$
- $\overline{H} = H/A$: $\overline{H} = SO_3$
- $F$: $H$-gauge theory
- $\tilde{F}$: $\overline{H}$-gauge theory
Example: gauge theory with $BA$-symmetry

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any dimension

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$BA$  
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$H$  
Lie group with $A \subset Z(H)$ $H = \text{SU}_2$

$\overline{H} = H/A$  
$\overline{H} = \text{SO}_3$

$F$  
$H$-gauge theory

$\tilde{F}$  
$\overline{H}$-gauge theory

A quotient construction allows to recover absolute $\overline{H}$-gauge theory as a sandwich (later)
Defects: quantum mechanics

$n = 1$

$\mathcal{H}$  Hilbert space

$H$  Hamiltonian

$G \subset \mathcal{H}$  finite group

Consider a point defect in $\mathcal{H}$. The link of a point in a 1-manifold (imaginary time) is $S^0$, a 0-sphere of radius $\varepsilon$, and the vector space of defects is

$$\lim_{\varepsilon \to 0} \text{Hom}_{\mathcal{H}}(S^0, \mathcal{H})$$

which is a space of singular operators on $\mathcal{H}$. To focus on formal aspects we write '$\text{End}_{\mathcal{H}}$'.

We now consider defects in $\mathcal{F}$, $\mathcal{G}$, $r$ which transport to point defects in $\mathcal{H}$. 

\[ \mathcal{C}(G) \quad \sigma \quad G \mathcal{G}(\mathcal{H}, \mathcal{H}) \quad (\mathcal{H}, \mathcal{H}) \]
Consider a point defect in $F$. The link of a point in a 1-manifold (imaginary time) is $S^0$, a 0-sphere of radius $\varepsilon$, and the vector space of defects is

$$\lim_{\varepsilon \to 0} \text{Hom}(1, F(S^0_\varepsilon))$$

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which is a space of singular operators on $\mathcal{H}$. To focus on formal aspects we write ‘End($\mathcal{H}$)’

We now consider defects in $(\rho, \sigma, \tilde{F})$ which transport to point defects in $F$
Point $\rho$-defects

The link is a closed interval with $\rho$-colored boundary. It evaluates under $(\sigma, \rho)$ to the vector space $A = \mathbb{C}[G]$. The “label” of the defect is therefore an element of $A$. Note $G \subset A$ labels invertible defects.

$\rho$-defects are topological

$\mathbb{C}(G) \quad \sigma \quad G_{G(\mathbb{H}, \mathbb{H})} \quad (\mathbb{H}, \mathbb{H}) \quad \simeq \quad A$
Point $\tilde{F}$-defects

The link is again a closed interval, but now with $\tilde{F}$-colored boundary. The value under $(\sigma, \tilde{F})$ is $\text{End}_A(\mathcal{H})$, the space of observables that commute with the $G$-action. $\tilde{F}$-defects are typically not topological.
Point $\sigma$-defects: central defects

The link is $S^1$, and the value under $\sigma$ is the vector space which is the center of the group algebra $A = \mathbb{C}[G]$.

$\sigma$-defects are topological
The general point defect

A general point defect in $F$ can be realized by a line defect in $(\rho, \sigma, \widetilde{F})$.

Label the defect beginning with the highest dimensional strata and work down in dimension:

- $B$ \((A, A)\)-bimodule
- $\xi$ vector in $B$
- $T$ \((A, A)\)-bimodule map $B \rightarrow \text{End}(\mathcal{H})$
Composition law on defects

Compute using the links of the defects—2 incoming and 1 outgoing

\(\sigma\)-defects: pair of pants

\(\rho\)-defects: pair of chaps
Commutation relations among defects

The sandwich realization makes clear that

- \( \rho \)-defects (symmetries) commute with \( \tilde{F} \)-defects
- \( \sigma \)-defects (central symmetries) commute with both \( \rho \)-defects and with \( \tilde{F} \)-defects

However, \( \rho \)-defects do not necessarily commute with each other nor do they commute with the general defect.
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- $\sigma$-defects (central symmetries) commute with both $\rho$-defects and with $\tilde{F}$-defects

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Commutation relations among defects

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However, \( \rho \)-defects do not necessarily commute with each other

Nor do they commute with the general defect
Finite group symmetries of an \((n = 2)\)-dimensional theory

Let \(G\) be a finite group, and let \(\sigma\) be the 3-dimensional finite \(G\)-gauge theory

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\sigma : \text{Bord}_3 \to \text{Alg}(\text{Cat})
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with \(\sigma(\text{pt}) = \text{Vect}[G]\), and let \(\rho\) be the regular right \(\sigma\)-module with \(\rho(\text{pt}) = \text{Vect}[G]_{\text{Vect}[G]}\).
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As opposed to \(G\)-symmetry in \(n = 1\), here the center is “bigger”
Quotients and quotient defects

We use the yoga of fully local topological field theory: let $\mathcal{C}'$ be a symmetric monoidal $n$-category and set $\mathcal{C} = \text{Alg}(\mathcal{C}')$, the $(n + 1)$-category whose objects are algebras in $\mathcal{C}'$. 
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**Definition:** An augmentation $\varepsilon_A : A \to 1$ of an algebra $A \in \text{Alg}(\mathcal{C}')$ is an algebra homomorphism from $A$ to the tensor unit $1 \in \mathcal{C}$.
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Augmentations are also called Neumann boundary theories
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Augmentations are also called *Neumann boundary theories*

Augmentations do not always exist
**Definition:** Suppose given finite symmetry data \((\sigma, \rho)\) and a \((\sigma, \rho)\)-module structure \((\tilde{F}, \theta)\) on a quantum field theory \(F\). Suppose \(\varepsilon\) is an augmentation of \(\sigma\). Then the *quotient* of \(F\) by the symmetry \(\sigma\) is

\[
\frac{F}{\sigma} = \varepsilon \otimes_{\sigma} \tilde{F}
\]
Dirichlet-to-Neumann and Neumann-to-Dirichlet domain walls

The categories of domain walls $\rho \rightarrow \epsilon$ and $\epsilon \rightarrow \rho$ are each free of rank one; let

$$\delta : \rho \longrightarrow \epsilon$$
$$\delta^* : \epsilon \longrightarrow \rho$$

be generators. Transporting via $\theta$ we obtain domain walls

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$$\delta : F \rightarrow F/\sigma$$
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We will soon compute the self-domain wall

$$\delta^* \circ \delta : F \rightarrow F$$
Quotient defects (after Roumpedakis–Seifnashri–Shao arXiv:2204.02407)

Passing from $F$ to $F/\sigma$ on a manifold $M$ places the topological defect $\epsilon$ on all of $M$
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There is also a quotient defect $\epsilon(Z)$—it is a $\rho$-defect—supported on a submanifold $Z \subset M$, defined using a tubular neighborhood $\nu$ of $Z \subset M$. It is topological, as are all $\rho$-defects.

\[
\text{Codim}_M(Z) = 1
\]

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If $\text{codim}_M(Z) = 1$, then

$$\epsilon(Z) = \delta^* \circ \delta$$
Computation for finite homotopy theories

*Finite homotopy theories* are a special class of topological field theories, introduced in 1988 by Kontsevich, picked up a few years later by Quinn, developed by Turaev, ...
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They occur often in this context as $\sigma = \sigma_X$, e.g., for $X = BG$ or $X = B^{p+1}A$ or extensions

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$$B^2A \longrightarrow \mathcal{X} \longrightarrow BG$$

Defects—in particular quotient defects—can be made explicit and computations are easy. Here is the composition $\delta^* \circ \delta$, essentially a finite homotopy theory based on $\Omega \mathcal{X}$:
I conclude with an application—symmetry used to constrain dynamics via:

If a gapped theory $F_{UV}$ has a $(\sigma, \rho)$-module structure, then the low energy topological field theory approximation $F_{IR}$ should also have a $(\sigma, \rho)$-module structure.

$$\downarrow \text{RG flow}$$

$$(\sigma, \rho) \subset F_{IR}$$

We will prove in a particular example that there does not exist a topological left $\tilde{F}$ such that $\rho_{\tilde{F}}$ is invertible. Therefore, $F_{UV}$ cannot flow to an invertible field theory, i.e., is not "trivially gapped."
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Warning: To apply to the following example, $\sigma$ here includes the duality defect $\Delta$
Duality defect

σ  \( n + 1 \)-dimensional topological field theory
ρ  right regular \( \sigma \)-module
ε  augmentation of \( \sigma \): “invertible” right \( \sigma \)-module
\( \tilde{F} \)  left \( \sigma \)-module
\( F \)  \( n \)-dimensional QFT \( \rho \otimes_\sigma \tilde{F} \)
\( F/\sigma \)  \( n \)-dimensional QFT \( \epsilon \otimes_\sigma \tilde{F} \)

\[ \sigma \] \( \sim \) \( F \) \[ \Theta \] \[ \sim \] \[ \sim \] \[ \sim \] \( \tilde{F} \) \[ \sim \] \[ \sim \] \[ \sim \] \[ \sim \] \( F/\sigma \)
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Suppose there is an isomorphism \( \phi: F/\sigma \xrightarrow{\sim} F \). Recall \( \delta: F \rightarrow F/\sigma \)
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**Definition:** The *duality defect* \( \Delta \) is the self-domain wall

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\Delta = \phi \circ \delta: F \longrightarrow F
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\[ \Delta = \phi \circ \delta : F \longrightarrow F \]

**Computation:** \( \Delta^* \circ \Delta = (\phi \delta)^*(\phi \delta) = \delta^* \phi^* \phi \delta = \delta^* \circ \delta \) since \( \phi^* = \phi^{-1} \) (\( \phi \) is invertible)
Example

Let $n = 4$ and let $\sigma$ be the 5-dimensional finite homotopy theory built from $X = B^2/\mu_2$.
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This models \( B/\mu_2 \)-symmetry ("1-form symmetry").

Recall that \( \rho, \epsilon, \delta, \) and \( \delta^* \) and the composition \( \delta^* \circ \delta \) fit into the diagram

\[
\begin{array}{ccc}
\Omega \mathcal{X} & \xrightarrow{\ast} & \mathcal{X} \\
\xleftarrow{\ast} & \xleftarrow{\ast} & \xleftarrow{\ast} \\
\mathcal{X} & \xrightarrow{\epsilon} & \mathcal{X} \\
\rho & \rho & \rho
\end{array}
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\( \delta^* \circ \delta \) is roughly 3-dimensional \( /\mu_2 \)-gauge theory.
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Recall that $\rho$, $\epsilon$, $\delta$, and $\delta^*$ and the composition $\delta^* \circ \delta$ fit into the diagram

$\delta^* \circ \delta$ is roughly 3-dimensional $\mu_2$-gauge theory

In an invertible $(\sigma, \rho)$-module $\lambda$, the self-domain wall $\delta^* \circ \delta$ is multiplication by 3-dimensional $\mu_2$-gauge theory.
Now suppose $F$ is a 4d QFT with a left $(\sigma, \rho)$-structure, and assume given an isomorphism

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**Example:** $F$ is $U_1$ gauge theory with coupling constant $\tau$

- $F$ has $B|\mu_2$ symmetry from $\mu_2 \subset U_1$
- $F/\sigma$ is $U_1$ gauge theory with coupling constant $\tau/4$
- $\phi$ is S-duality which sends $\tau \mapsto -1/\tau$
- Set $\tau = 2\sqrt{-1}$
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**Theorem:** No such square root exists
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**Theorem:** No such square root exists

**Conclusion:** The gauge theory $F$ is not trivially gapped
Notes from a set of four summer school lectures on this topic are at

https://web.ma.utexas.edu/users/dafr/Freed_perim.pdf

and (very soon) on the Global Categorical Symmetries website:

https://scgcs.berkeley.edu/2022-school/

The latter has lecture notes on related topics and there are more resources at:

https://scgcs.berkeley.edu/