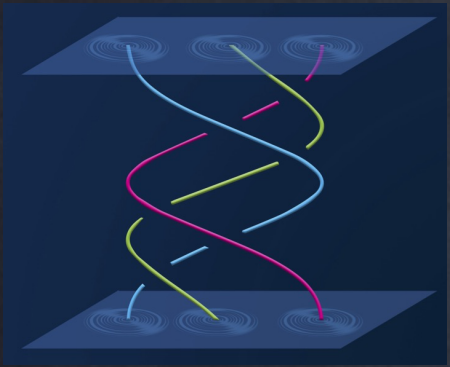
Finite Symmetry in Field Theory

Dan Freed
University of Texas at Austin

July 12, 2022

Joint work with Greg Moore and Constantin Teleman

Simons Collaboration (https://scgcs.berkeley.edu)



Global Categorical Symmetry

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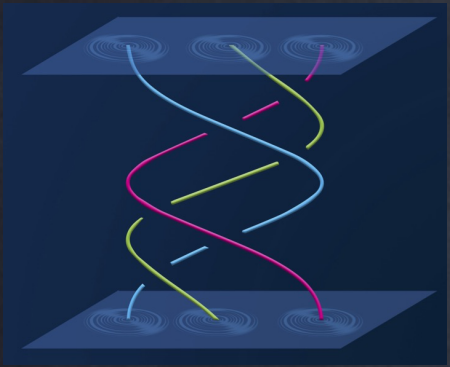

Categorical Symmetry

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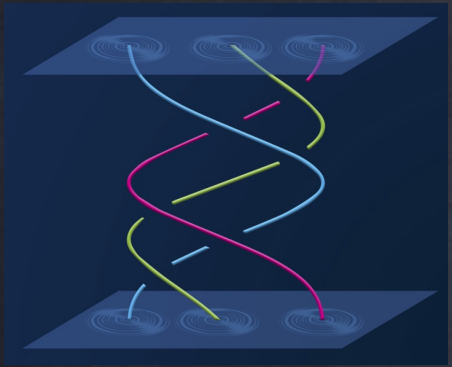

Symmetry

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Symmetry in QFT

Motivating idea for this talk:

Separate out the abstract structure of symmetry from its concrete manifestations as actions or representations

Symmetry in QFT is a big topic; the framework discussed here only scratches the surface

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Many current results about extended notions of symmetry in QFT: Apruzzi, Bah, Benini, Bhardwaj, Bonetti, Bullimore, Córdova, Choi, Cvetič, Del Zotto, Dumitrescu, Gaiotto, García Etxebarria, Gould, Gukov, Heckman, Heidenreich, Hopkins, Hosseini, Hsin, Hübner, Intriligator, Ji, Jian, Johnson-Freyd, Jordan, Kaidi, Kapustin, Komargodski, Lake, Lam, McNamara, Minasian, Montero, Ohmari, Pantev, Pei, Plavnik, Reece, Robbins, Roumpedakis, Rudelius, Schäfer-Nameki, Scheimbauer, Seiberg, Seifnashri, Shao, Sharpe, Tachikawa, Thorngren, Torres, Vandermeulen, Wang, Wen, Willett, ..., ..., ...

Motivation: algebras

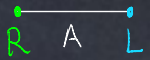
Abstract symmetry data (for algebras) is a pair (A, R):

A algebra

R right regular module

Definition: Let V be a vector space. An (A, R)-*action on* V is a pair (L, θ) consisting of a left A-module L together with an isomorphism of vector spaces

$$\theta\colon R \otimes_A L \xrightarrow{\cong} V$$



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Analogy:

algebra $\rightsquigarrow$ topological field theory

element of algebra $\rightsquigarrow$ defect in TFT

Example: Let G be a finite group. Its group algebra is

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} \lambda_g \, g \right\}, \qquad \lambda_g \in \mathbb{C}$$

Identify $\mathbb{C}[G] = \mathrm{Fun}(G)$; convolution product is pushforward under

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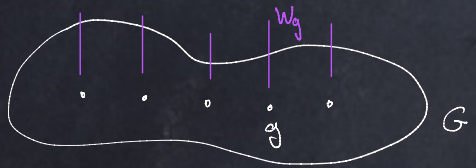
Higher Example: Vect = category of finite dimensional complex vector spaces. Define $\mathrm{Vect}[G]$ as the linear category (Vect-module) of vector bundles over G with tensor product pushforward under mult. It is a *fusion category*



$$(W_1 * W_2)_g = \bigoplus_{g_1 g_2 = g} (W_1)_{g_1} \otimes (W_2)_{g_2}$$

Quotients: augmentations

Definition: An *augmentation* of an algebra A is an algebra homomorphism $\epsilon\colon A \to \mathbb{C}$.

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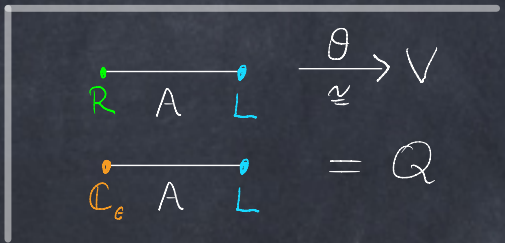

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Augmentations for higher algebras: Φ tensor category $\epsilon\colon \Phi \to \mathrm{Vect}$ *fiber functor*

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Isomorphism class of extension $[\tau] \in H^2(G;\mathbb{C}^\times)$

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An augmentation $\epsilon \colon A^\tau \to \mathbb{C}$ splits the extension, so does not exist if $[\tau] \neq 0$

Field theory

Analogy: field theory $\sim$ module over an algebra OR $\sim$ representation of a Lie group

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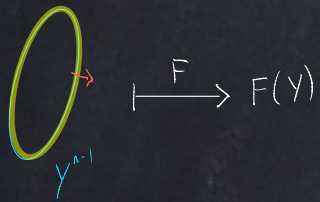
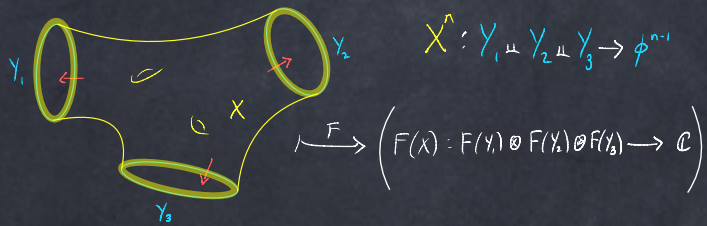
n	dimension of spacetime
$\mathcal{F}$	background fields (orientation, Riemannian metric, ...)





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Kontsevich-Segal: Axioms for 2-tier nontopological theory $F\colon \mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{F}) \to t\,\mathrm{Vect}$

Domain walls, boundary theories, defects

$\sigma, \sigma_1, \sigma_2$	$(n+1)$-dimensional theories
$\delta\colon\ \sigma_1 \to \sigma_2$	domain wall
$\rho\colon\ \sigma \to \mathbb{1}$	right boundary theory
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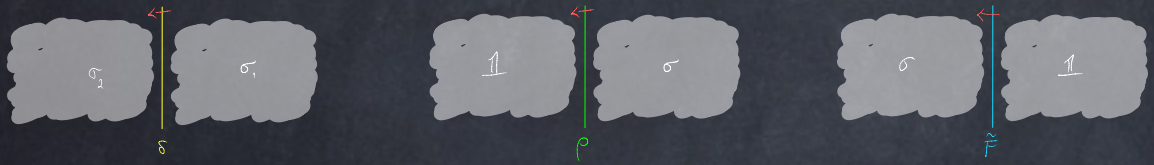

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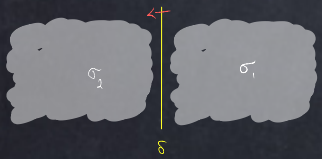

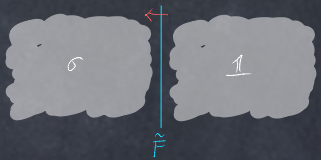

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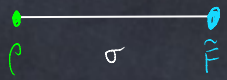

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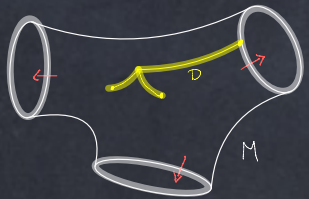
More generally, one can put *defects* on any (stratified) manifold $D \subset M$



Composition laws; invertibility

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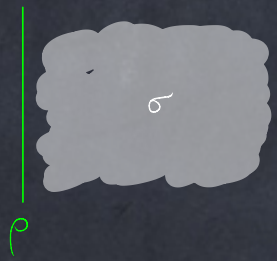
So a notion of *invertible* field theory and *invertible* defect

Main definition: abstract symmetry data

Fix a dimension n and background fields $\mathcal{F}$ (which we keep implicit)

Definition: *Finite field-theoretic symmetry data of dimension* n is a pair (σ, ρ) in which σ is an $(n+1)$-dimensional <u>topological</u> field theory and ρ is a <u>topological</u> right σ-module.



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A regular boundary theory is also called *Dirichlet*

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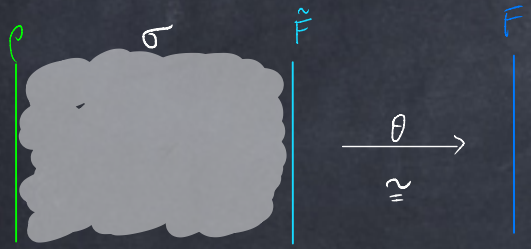

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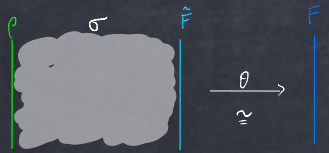

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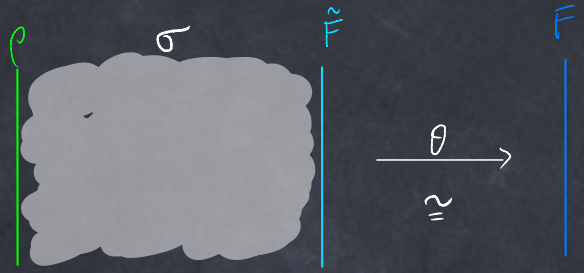
- Symmetry persists under renormalization group flow, hence a low energy approximation to F should also be an (σ,ρ)-module. If F is gapped, then we can bring to bear powerful methods and theorems in topological field theory to investigate *topological* left σ-modules. This leads to dynamical predictions

Example: quantum mechanics with G-symmetry

$n = 1$

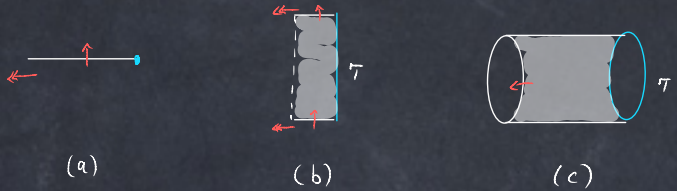
$\mathcal{F}$	{orientation, Riemannian metric} for F and $\tilde{F}$
$\mathcal{H}$	Hilbert space
H	Hamiltonian
$G \circlearrowright \mathcal{H}$	finite group
$S\colon G \to \mathrm{Aut}(\mathcal{H})$	action on $\mathcal{H}$
$\sigma(\mathrm{pt})$	$\mathbb{C}[G]$
$F(\mathrm{pt})$	$\mathcal{H}$
$\tilde{F}(\mathrm{pt})$	${}_{\mathbb{C}[G]}\mathcal{H}$ (left module)



Example: quantum mechanics with G-symmetry

$n = 1$

$\mathcal{F}$	{orientation, Riemannian metric} for F and $\widetilde{F}$
$\mathcal{H}$	Hilbert space
H	Hamiltonian
$G \circlearrowright \mathcal{H}$	finite group
$S\colon G \to \mathrm{Aut}(\mathcal{H})$	action on $\mathcal{H}$
$\sigma(\mathrm{pt})$	$\mathbb{C}[G]$
$F(\mathrm{pt})$	$\mathcal{H}$
$\widetilde{F}(\mathrm{pt})$	${}_{\mathbb{C}[G]}\mathcal{H}$ (left module)



Evaluation of some bordisms:

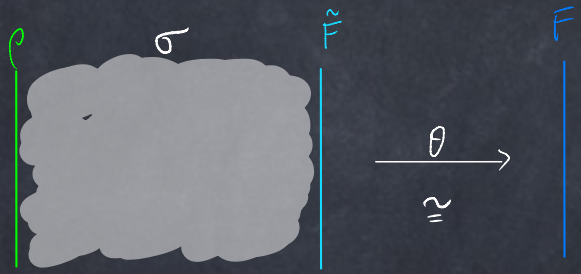
(a) the left module ${}_{\mathbb{C}[G]}\mathcal{H}$

(b) $e^{-\tau H/\hbar}\colon {}_{\mathbb{C}[G]}\mathcal{H} \longrightarrow {}_{\mathbb{C}[G]}\mathcal{H}$

(c) the central function $g \longmapsto \mathrm{Tr}_{\mathcal{H}}\big(S(g)e^{-\tau H/\hbar}\big)$ on G

Example: gauge theory with BA-symmetry

n	any dimension
A	finite abelian group $A = \mu_2$
BA	a homotopical/shifted A ("1-form A-symmetry")
H	Lie group with $A \subset Z(H)$ $H = \mathrm{SU}_2$
$\overline{H} = H/A$	$\overline{H} = \mathrm{SO}_3$
F	H-gauge theory
$\tilde{F}$	$\overline{H}$-gauge theory



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A quotient construction allows to recover absolute $\overline{H}$-gauge theory as a sandwich (later)

Defects: quantum mechanics

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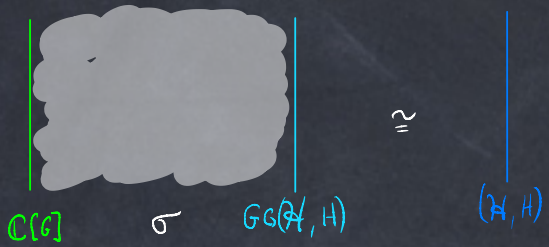

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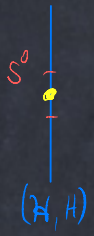

Consider a point defect in F. The link of a point in a 1-manifold (imaginary time) is S^0, a 0-sphere of radius ϵ, and the vector space of defects is

$$\varprojlim_{\epsilon \to 0} \mathrm{Hom}\big(1, F(S^0_\epsilon)\big)$$

which is a space of singular operators on $\mathcal{H}$. To focus on formal aspects we write '$\mathrm{End}(\mathcal{H})$'

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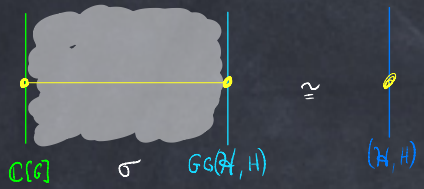

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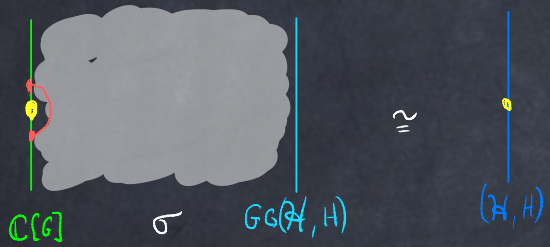
We mow consider defects in $(\rho, \sigma, \widetilde{F})$ which transport to point defects in F

Point ρ-defects

The link is a closed interval with ρ-colored boundary. It evaluates under (σ, ρ) to the *vector space* $A = \mathbb{C}[G]$. The "label" of the defect is therefore an element of A. Note $G \subset A$ labels invertible defects.

ρ-defects are topological

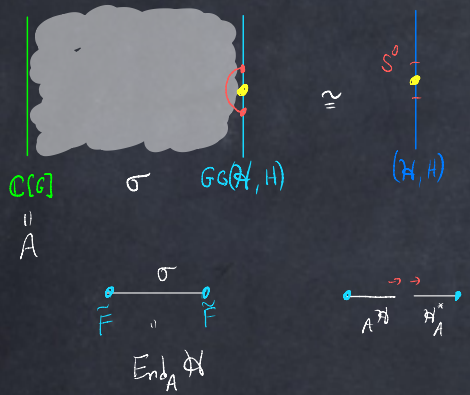


$\cong A$

Point $\widetilde{F}$-defects

The link is again a closed interval, but now with $\widetilde{F}$-colored boundary. The value under $(\sigma, \widetilde{F})$ is $\mathrm{End}_A(\mathcal{H})$, the space of observables that commute with the G-action

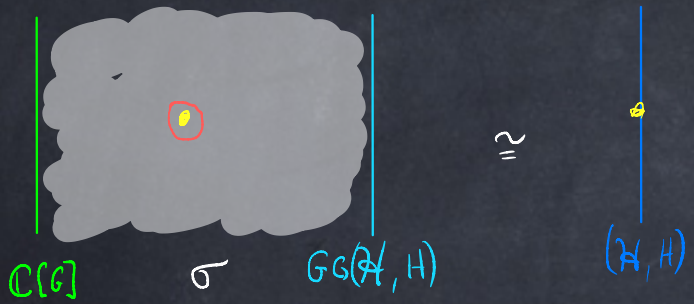
$\widetilde{F}$-defects are typically not topological



Point σ-defects: central defects

The link is S^1, and the value under σ is the vector space which is the center of the group algebra $A = \mathbb{C}[G]$.

σ-defects are topological

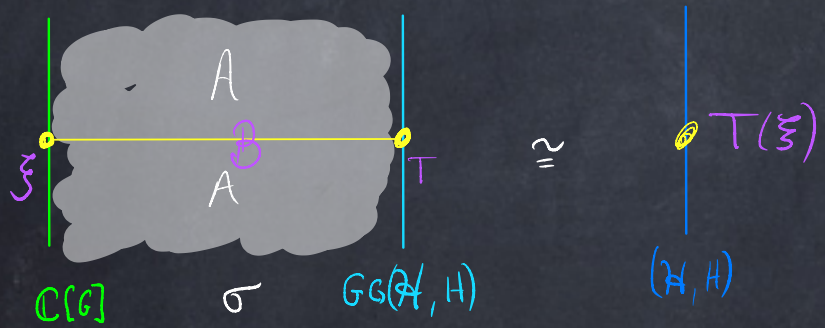


The general point defect

A general point defect in F can be realized by a line defect in $(\rho, \sigma, \widetilde{F})$.

Label the defect beginning with the highest dimensional strata and work down in dimension

B	(A, A)-bimodule
ξ	vector in B
T	(A, A)-bimodule map $B \longrightarrow \mathrm{End}(\mathcal{H})$

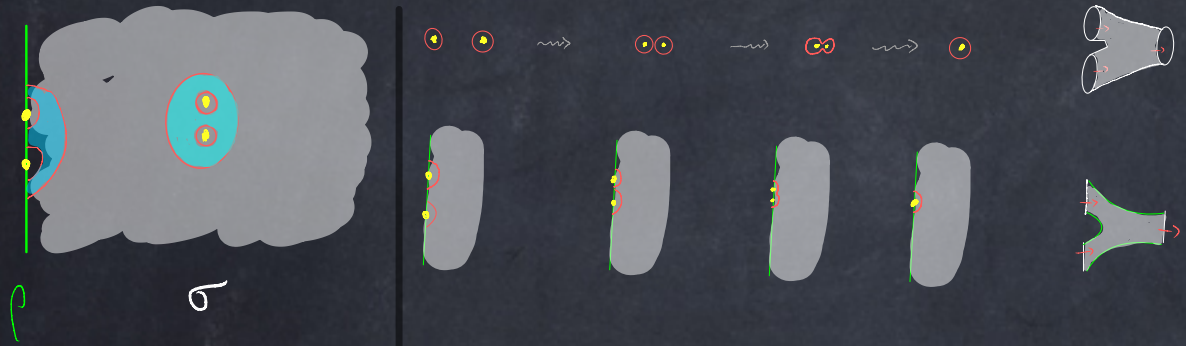


Composition law on defects

Compute using the links of the defects—2 incoming and 1 outgoing

σ-defects: pair of pants

ρ-defects: pair of chaps



Commutation relations among defects

The sandwich realization makes clear that

- ρ-defects (symmetries) commute with $\widetilde{F}$-defects
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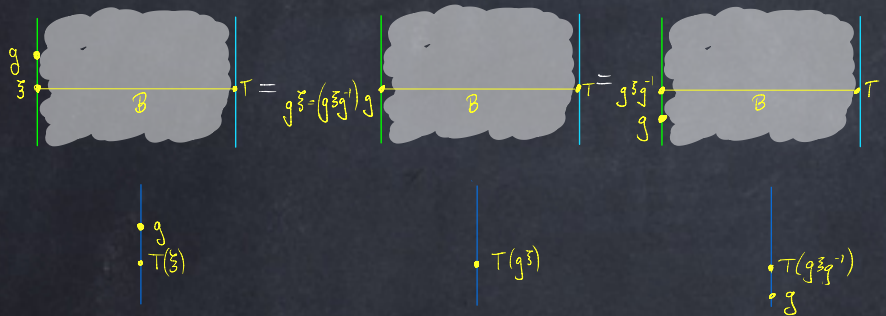
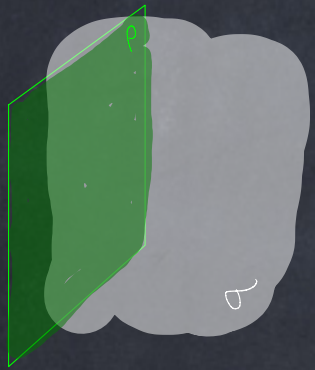


Finite group symmetries of an $(n = 2)$-dimensional theory

Let G be a finite group, and let σ be the 3-dimensional finite G-gauge theory

$$\sigma\colon\ \mathrm{Bord}_3 \longrightarrow \mathrm{Alg}(\mathrm{Cat})$$

with $\sigma(\mathrm{pt}) = \mathrm{Vect}[G]$, and let ρ be the regular right σ-module with $\rho(\mathrm{pt}) = \mathrm{Vect}[G]_{\mathrm{Vect}[G]}$



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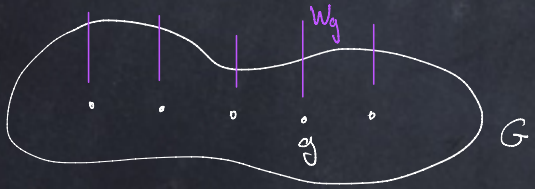

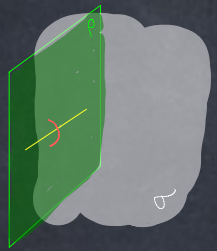

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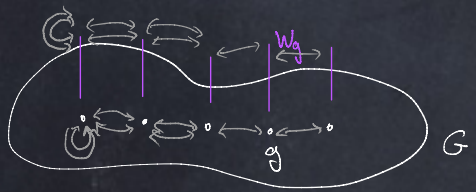

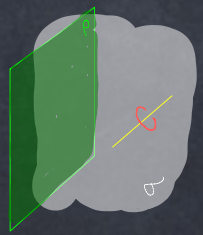

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As opposed to G-symmetry in $n = 1$, here the center is "bigger"

Quotients and quotient defects

We use the yoga of fully local topological field theory: let $\mathcal{C}'$ be a symmetric monoidal n-category and set $\mathcal{C} = \mathrm{Alg}(\mathcal{C}')$, the $(n+1)$-category whose objects are algebras in $\mathcal{C}'$

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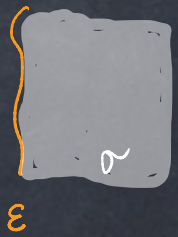
Augmentations are also called *Neumann boundary theories*



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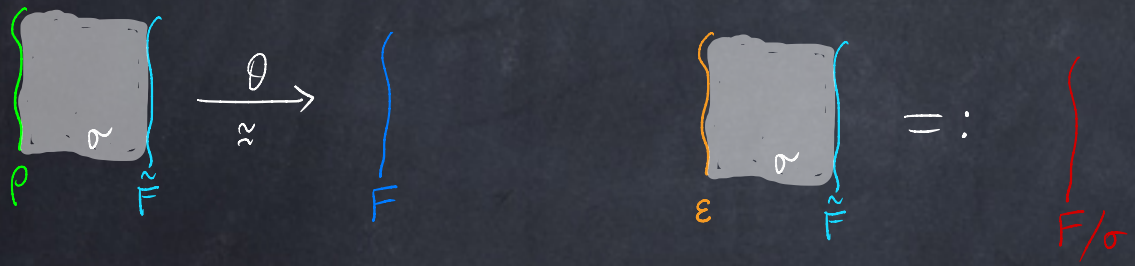
Augmentations do not always exist

Definition: Suppose given finite symmetry data (σ, ρ) and a (σ, ρ)-module structure $(\widetilde{F}, \theta)$ on a quantum field theory F. Suppose ϵ is an augmentation of σ. Then the *quotient* of F by the symmetry σ is

$$F\big/_{\epsilon}\sigma = \epsilon \otimes_{\sigma} \widetilde{F}$$



Dirichlet-to-Neumann and Neumann-to-Dirichlet domain walls

The categories of domain walls $\rho \to \epsilon$ and $\epsilon \to \rho$ are each free of rank one; let

$$\delta : \rho \longrightarrow \epsilon$$
$$\delta^* : \epsilon \longrightarrow \rho$$

be generators. Transporting via θ we obtain domain walls

$$\delta : F \longrightarrow F/\sigma$$
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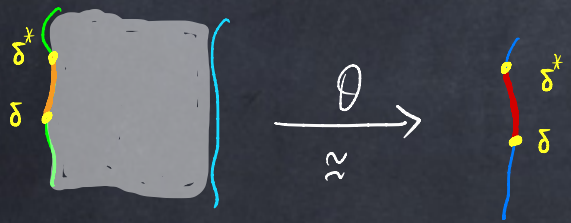

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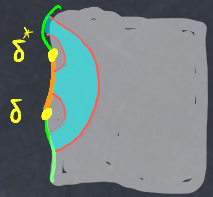

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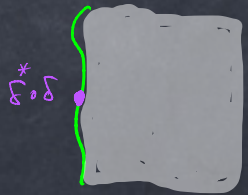

We will soon compute the self-domain wall

$$\delta^* \circ \delta : F \longrightarrow F$$

Quotient defects (after Roumpedakis–Seifnashri–Shao arXiv:2204.02407)

Passing from F to F/σ on a manifold M places the topological defect ϵ on all of M

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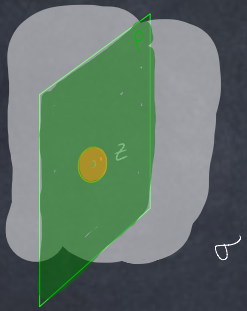
$\mathrm{Codim}_M(Z) = 1$



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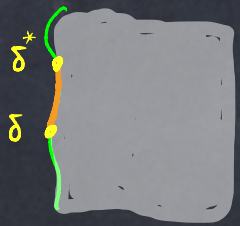
If $\mathrm{codim}_M(Z) = 1$, then

$$\epsilon(Z) = \delta^* \circ \delta$$



Computation for finite homotopy theories

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They occur often in this context as $\sigma = \sigma_{\mathcal{X}}$, e.g., for $\mathcal{X} = BG$ or $\mathcal{X} = B^{p+1}A$ or extensions

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Defects—in particular quotient defects—can be made explicit and computations are easy. Here is the composition $\delta^* \circ \delta$, essentially a finite homotopy theory based on $\Omega\mathcal{X}$:

$$\begin{tikzcd}
 & & \Omega\mathcal{X} \arrow[dl, dashed] \arrow[dr, dashed] & & \\
 & * \arrow[dl] \arrow[dr] & & * \arrow[dl] \arrow[dr] & \\
* \arrow[drr, "\rho"'] & & \mathcal{X} \arrow[d, "\epsilon"] & & * \arrow[dll, "\rho"] \\
 & & \mathcal{X} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
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* \arrow[dr, "\rho"'] & & * \arrow[dl, "\rho"] \\
 & \mathcal{X} &
\end{tikzcd}$$

Duality defects (after Córdova–Choi–Hsin–Lam–Shao arXiv:2111.01139)

I conclude with an application—symmetry used to constrain dynamics via:

> *If a gapped theory F_{UV} has a (σ, ρ)-module structure, then the low energy topological field theory approximation F_{IR} should also have a (σ, ρ)-module structure*

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We will prove in a particular example that there does not exist a *topological* left σ-module $\tilde{\lambda}$ such that $\lambda := \rho \otimes_\sigma \tilde{\lambda}$ is invertible. Therefore, F_{UV} cannot flow to an invertible field theory, i.e., is not "trivially gapped"

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It follows that $A := \sigma(\mathrm{pt})$ does not admit an augmentation ('t Hooft anomaly)

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I conclude with an application—symmetry used to constrain dynamics via:

> *If a gapped theory F_{UV} has a (σ, ρ)-module structure, then the low energy topological field theory approximation F_{IR} should also have a (σ, ρ)-module structure*

$$\begin{tikzcd} (\sigma, \rho) \circlearrowright F_{\mathrm{UV}} \arrow[d, squiggly, "\text{RG flow}"] \\ (\sigma, \rho) \circlearrowright F_{\mathrm{IR}} \end{tikzcd}$$

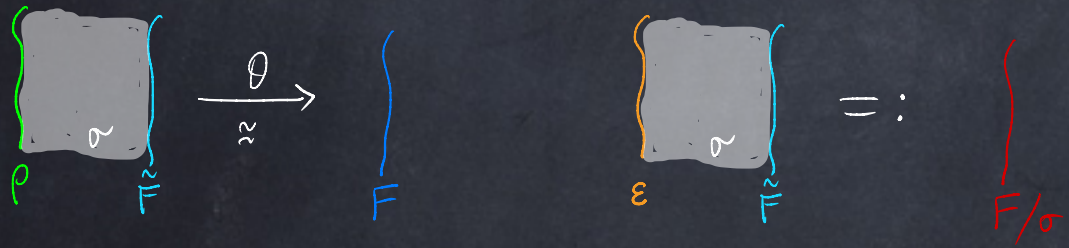
We will prove in a particular example that there does not exist a *topological* left σ-module $\tilde{\lambda}$ such that $\lambda := \rho \otimes_\sigma \tilde{\lambda}$ is invertible. Therefore, F_{UV} cannot flow to an invertible field theory, i.e., is not "trivially gapped"

It follows that $A := \sigma(\mathrm{pt})$ does not admit an augmentation ('t Hooft anomaly)

Warning: To apply to the following example, σ here includes the duality defect Δ

Duality defect

σ	$n+1$-dimensional topological field theory
ρ	right regular σ-module
ϵ	augmentation of σ: "invertible" right σ-module
$\widetilde{F}$	left σ-module
F	n-dimensional QFT $\rho \otimes_\sigma \widetilde{F}$
F/σ	n-dimensional QFT $\epsilon \otimes_\sigma \widetilde{F}$



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Computation: $\Delta^* \circ \Delta = (\phi\delta)^*(\phi\delta) = \delta^*\phi^*\phi\delta = \delta^* \circ \delta$ since $\phi^* = \phi^{-1}$ (ϕ is invertible)

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Recall that ρ, ϵ, δ, and δ^* and the composition $\delta^* \circ \delta$ fit into the diagram

$$\begin{tikzcd}
 & & \Omega\mathcal{X} \arrow[dl, dashed] \arrow[dr, dashed] & & \\
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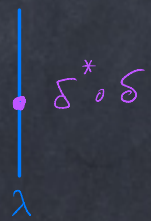

In an invertible (σ, ρ)-module λ, the self-domain wall $\delta^* \circ \delta$
is multiplication by 3-dimensional μ_2-gauge theory

Now suppose F is a 4d QFT with a left (σ, ρ)-structure, and assume given an isomorphism

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Conclusion: The gauge theory F is not trivially gapped

Notes from a set of four summer school lectures on this topic are at

https://web.ma.utexas.edu/users/dafr/Freed_perim.pdf

and (very soon) on the Global Categorical Symmetries website:

https://scgcs.berkeley.edu/2022-school/

The latter has lecture notes on related topics and there are more resources at:

https://scgcs.berkeley.edu/