Geometry of Higher-Form Structures in String Theory

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Motivation

Studying M/string theory on special holonomy spaces $X$:

- Non-compact spaces $X \rightarrow$
  Geometric engineering of supersymmetric quantum field theories (SQFTs):
    - Build dictionary:
      - $\{\text{operators, symmetries}\} \leftrightarrow \{\text{geometry, topology}\}$

Focus: higher-form global symmetries
(associated with "flavor" branes)

- Compact spaces $X \rightarrow$
  Quantum field theory (QFT) w/ gravity $\rightarrow$
  Higher-form symmetries gauged or broken
  [Physical consistency conditions: swampland program]
Higher-form symmetries in (S)QFT - active field of research

[Gaiotto, Kapustin, Seiberg, Willet, 2014],...
c.f., talks by Lakshya Bhardwaj, Dan Freed

Higher-form symmetries & geometric engineering

[Del Zotto, Heckman, Park, Rudelius, 2015],...

[Morrison, Schäfer-Nameki, Willett, 2020],
[Albertini, Del Zotto, Garcia Etxebarria, Hosseini, 2020],...

[Del Zotto, Ohmori, 2020],[Apruzzi, Dierigl, Lin, 2020],
[M.C., Dierigl, Lin, Zhang, 2020],..

[Apruzzi,Bhardwaj, Oh, Schäfer-Nameki, 2021],
[Apruzzi, Bhardwaj, Gould, Schäfer-Namek, 2021],
[Bhardwaj, Giacomelli, Hübner, Schäfer-Nameki, 2021],
[M.C., Dierigl, Lin, Zhang, 2021],...

[Del Zotto,Heckman, Meynet, Moscrop, Zhang, 2022],

[M.C., Heckman, Hübner, Torres, 2022],
[Del Zotto, Garcia Etxebarria, Schäfer-Nameki, 2022],

[Hübner, Morrison, Schäfer-Nameki,Wang, 2022]
[Heckman, Lawrie, Lin, Zhang, Zoccarato, 2022],...
c.f., also talks by Muyang Liu, Paul Oehlmann in parallel session
Goals

- Identify geometric origin of higher-form symmetries for M-theory non-compact special holonomy spaces $X$

  Punchline: 0-form, 1-form and 2-group symmetries via cutting and gluing of singular boundary geometries

- Examples: M-theory on non-compact elliptically fibered Calabi-Yau n-folds [building on toric orbifold results]

  A few statements about compact examples [dual to F-theory compactification]
Based on

- M. C., J. J. Heckman, M. Hübner and E. Torres, "0-Form, 1-Form and 2-Group Symmetries via Cutting and Gluing of Orbifolds," 2203.10102 non-compact examples & work to appear compact examples

Related: Gauge symmetry topology constraints in D=8 N=1SG


- "One Loop to Rule Them All: Eight and Nine Dimensional String Vacua from Junctions,” arXiv:2203.03644 [hep-th]- String junctions
Outline:

• Introduction

• Defect group and higher-form symmetries → Topology of flavor symmetry group

• 2-group symmetries

• Compact (elliptic) examples → fate of higher-form symmetries

• Concluding remarks
I. Introduction

Defect group for M-theory on non-compact X

- Defect Group for extended $p$-dim operators associated with M2 and M5 branes: $\mathcal{D}_p = \mathcal{D}_p^{M2} \oplus \mathcal{D}_p^{M5}$

- M2, M5 in X live on relative cycles:
  
  \begin{align*}
  \mathcal{D}_p^{M2} &= \frac{H_{3-p}(X, \partial X)}{H_{3-p}(X)} \cong H_{3-p-1}(\partial X)|_{\text{triv}} [p\text{-dim el. operators in SCFT}] \\
  \mathcal{D}_p^{M5} &= \frac{H_{6-p}(X, \partial X)}{H_{6-p}(X)} \cong H_{6-p-1}(\partial X)|_{\text{triv}} [p\text{-dim mag. operators in SCFT}] 
  \end{align*}

- Defect pairing $\langle \cdot, \cdot \rangle$ in s.t. $\leftrightarrow$ Linking Pairing $\ell(\cdot, \cdot)$ in $\partial X$
Example: non-compact $K3$

[M.C., Dierigl, Lin, Zhang, 2021, 2022]

- Local elliptically fibered $K3$ \[ E \hookrightarrow X \rightarrow \mathbb{C} \]
- Singular fiber of Kodaira type $\phi$ at $z \in \mathbb{C}$ w/ monodromy $M$

- Boundary $\partial X \to S^1$
- Exact sequence for spaces fibered over circles:

\[
0 \to \text{coker}(M_n - 1) \to H_n(X) \to \ker(M_{n-1} - 1) \to 0
\]

$M_n$ - monodromy in homology in degree $n$

- $D_1^{M2} = D_4^{M5} = H_2(X, \partial X)/H_2(X) \cong \text{Tor Coker}(M - 1) = \langle \mathcal{F} \rangle$

- $X$ engineers 7D SYM with gauge algebra $g_{\phi}$ w/
Defect group $\mathcal{D} = \langle \mathcal{F} \rangle_1^{M2} \oplus \langle \mathcal{F} \rangle_4^{M5}$
Example (continued):

- Non-trivial self-linking/intersection: \( \ell(\partial \Sigma, \partial \Sigma) = \Sigma \cdot \Sigma \neq 0 \)

- Elements of \( D_1^{M2}, D_4^{M5} \) typically mutually non-local

- Choose electric polarization \( D_1^{M2} \) [for the rest of the talk]

- Gauge group is simply connected \( G_\phi \) w/algebra \( g_\phi \) (ADE)

- Resulting 7D SYM theory w/ gauge group \( G_\phi \)

- (Wilson) Line operators \( D_1^{M2} \) acted on by 1-form symmetry \( Z_{G_\phi} \) [Pontyagin dual to line operators] fixes group topology
Now, turn to higher-form structures for non-compact spaces $X$ in higher dimensions ($D \geq 6$)

→ leads to new phenomena
II. Geometrizing Topology of Flavor Group

Non-compact ADE loci $\equiv$ flavor branes $\rightarrow$ flavor symmetries

Naïve flavor symmetry $\widetilde{G}_F$ (simply connected w/ Lie Algebra $\mathfrak{g}_i$)

[From now on only ADE’s in $\partial X$]

$\widetilde{G}_F = \widetilde{G}_1 \times \widetilde{G}_2 \times \widetilde{G}_3 \times \ldots$

(Flavor Wilson) lines $\rightarrow$ fix topology of flavor symmetry $G_F$

from singular boundary topology
Boundary geometry of flavor branes:

- Singular non-compact space $X$ w/ $K=\bigcup_i K_i$ - ADE loci (of flavor branes) in the boundary $\partial X$
- Define a smooth boundary $\partial X^o = \partial X \setminus K \&$ a tubular region $T_K$ (excise $K$)
- Locally $T_K \cap \partial X^o \cong \bigcup_i K_i \times S^3 / \Gamma_i$

Naïve flavor center symmetry:

$$Z_{\tilde{G}_F} = \text{Tor} \ H_1(T_K \cap \partial X^o) \cong Z_{\tilde{G}_1} \oplus Z_{\tilde{G}_2} \oplus Z_{\tilde{G}_3} \oplus \ldots$$
Boundary geometry of true flavor center symmetry $Z_{GF}$

Key: Mayer-Vietoris sequence in homology for singular boundary

\[ \partial X = \partial X^\circ \cup T_K \]

\[
\begin{array}{cccccc}
0 & \to & \ker(\nu_1) & \to & H_1(\partial X^\circ \cap T_K) & \xrightarrow{\nu_1} & H_1(\partial X^\circ) \oplus H_1(T_K) & \to & H_1(\partial X) & \to & 0
\end{array}
\]

\[
Z_{GF} = \text{Ker} \left( \nu_1 : Z_{\tilde{G}_F} \cong H_1(\partial X^\circ \cap T_K) \to H_1(\partial X^\circ) \oplus H_1(T_K) \right)
\]

Motivated by orbifold homology:

\[ H_1(\partial X^\circ) = H_1^{\text{orb}}(\partial X) \]

[Thurston, 1980], [Moerdijk, Pronk, 1997]
Example: Elliptically fibered threefold

\[ E \hookrightarrow X_3 \rightarrow B = \mathbb{C}^2 \]  
(non-compact base)

\[ [(G_L; G_R)-Conformal Matter] \]

[Del Zotto, Heckman, Tomasiello, Vafa, 2014]

- Non-compact discriminant loci: \( \phi_L \) on \( \mathbb{C} \times \{0\} \) & \( \phi_R \) on \( \{0\} \times \mathbb{C} \)

- Boundary five-manifold

\[ E \hookrightarrow \partial X_3 \rightarrow S^3 \]

- Discriminant locus consists of two linking circles \( S^1_L \) & \( S^1_R \)

- Excise singular fibers

\[ E \hookrightarrow \partial X^o \rightarrow S^1_L \times S^1_R \]
Example (continued):

- Again, exact sequence for spaces fibered over circles $S^1_{LR}$

$$0 \to \text{coker}(M_{n-1}) \to H_n(X) \to \ker(M_{n-1} - 1) \to 0$$

$M_n$ - monodromy in homology in degree $n$.

- From monodromies $M_L$ & $M_R$ about $S^1_L$ & $S^1_R$:

$$\text{Tor} H_1(\partial X^\circ) = \text{Tor} \frac{\mathbb{Z}^2}{\text{Im}(M_{\Phi_L} - 1) \cup \text{Im}(M_{\Phi_R} - 1)}$$

- $(SU(n); SU(m))$ Conformal Matter

$$G_F = \frac{SU(n) \times SU(m)}{\mathbb{Z}_{\gcd(n,m)}}$$

- General: $G_F = \frac{G_L \times G_R}{\mathbb{Z}_{\text{diag}}}$
General Elliptically fibered Calabi-Yau n-folds

- $E \hookrightarrow X_n \to B_{n-1}$

- Focus on non-compact discriminant locus $\Delta$
  
  [Hübner, Morrison, Schäfer-Nameki, Wang 2022]

- Homology groups of $\partial X$, $\partial X^o$ & $\partial X_F = \partial X \setminus \{\text{singular fibers}\}$

- Deformation retractions of $\partial B \setminus \Delta$ lift to $\partial X_F$

- Glue singular fibers back in

  $\text{Tor } H_1(\partial X^o) = \text{Tor } H_1(\partial X_F) \oplus \text{Tor } H_1(\partial B)$

$\Rightarrow$ Further explicit elliptic examples; also, orbifolds, $G_2$, No time
III. 2-Groups

- Two key short exact sequence
  \[
  0 \to \mathcal{C} \to Z_{G_F} \to Z_{GF} \to 0 \\
  0 \to \mathcal{C}^\vee \to \tilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee \to 0
  \]

  [Postnikov class]

- \(Z_{GF}\): Center flavor symmetry
- \(Z_{\tilde{G}_F}\): Naïve center flavor symmetry
- \(\mathcal{A}^\vee\): Line operators modulo screening by local operators
- \(\tilde{\mathcal{A}}^\vee\): (Naïve) Line operators modulo screening by local operators transforming in reps. of \(Z_{GF}\)
- \(\mathcal{C}^\vee\): Line operators in the kernel of \(\tilde{\mathcal{A}}^\vee \to \mathcal{A}^\vee\)

Pontryagin duals: \(\mathcal{D}^\vee = \text{Hom}(\mathcal{D}, U(1))\); [Line operators \(\leftrightarrow\) One-forms]
• Taking the Pontryagin dual (one-form symmetries) of line-operators in the second sequence & reversing arrows:

\[ \begin{align*}
\mathbb{Z}_0\text{-form:} & \quad 0 \rightarrow \mathcal{C} \rightarrow \mathbb{Z}_{\tilde{G}} \rightarrow \mathbb{Z}_G \rightarrow 0 \\
\text{1-form:} & \quad 0 \rightarrow \mathcal{A} \rightarrow \tilde{\mathcal{A}} \rightarrow \mathcal{C} \rightarrow 0
\end{align*} \]

\[ \text{w/ Postnikov class } \quad P \in H^3(BG, \mathcal{A}) \]

\[ \text{[obtained via first extension } \quad w_2 \in H^2(BZ_{\tilde{G}}, \mathcal{C}) \& \quad \text{Bockstein homomorphis } \quad \beta : H^2(BZ_G, \mathcal{C}) \rightarrow H^3(BZ_G, \mathcal{A}) ] \]

• Sufficient: when second sequence does not split, i.e., \( \tilde{\mathcal{A}} \neq \mathcal{A} \oplus \mathcal{C} \)

\[ \rightarrow 2\text{-group} \]

• One can collapse the two sequences:

\[ \begin{align*}
\text{2-group:} & \quad 0 \rightarrow \mathcal{A} \rightarrow \tilde{\mathcal{A}} \rightarrow \mathbb{Z}_{\tilde{G}_F} \rightarrow \mathbb{Z}_{G_F} \rightarrow 0
\end{align*} \]
2-groups and Mayer-Vietoris

• Derive parallel homology sequences (physically motivated):

\[
0 \rightarrow \ker(\nu_1) \rightarrow H_1(\partial X^o \cap T_K) \xrightarrow{\nu_1} \frac{H_1(\partial X^o \cap T_K)}{\ker(\nu_1)} \rightarrow 0,
\]

\[
0 \rightarrow \frac{H_1(\partial X^o \cap T_K)}{\ker(\nu_1)} \rightarrow H_1(\partial X^o) \oplus H_1(T_K) \rightarrow H_1(\partial X) \rightarrow 0
\]

• Precisely the 2-group sequences!

• Again, when the bottom sequence does not split → 2-group mixing 0-form and 1-form (flavor) symmetries
IV. Compact Models

- Compact singular space $X \rightarrow$ theory that includes quantum gravity & global symmetries gauged or broken
- What is M-theory gauge group?
- Elliptically fibered geometries (via M/F-theory duality):
  - Non-Abelian group algebras – ADE Kodaira classification
    Group topology $\rightarrow$ Mordell-Weil torsion
    \[ [\text{Aspinwall, Morrison, 1998}],
    [\text{Mayrhofer, Morrison, Till, Weigand, 2014}], [\text{M.C., Lin, 2017}] \]
  - Abelian groups $\rightarrow$ Mordell-Weil ``free” part
    \[ [\text{Morrison, Park 2012}], [\text{M.C., Klevers, Piragua, 2013}],
    [\text{Borchmann, Mayrhofer, Palti, Weigand, 2013}]... \]
  - Total group topology $\rightarrow$ Shoida map of Mordell-Weil
    \[ \frac{U(1)^r \times G_{\text{non-ab}}}{\prod_{i=1}^r \mathbb{Z}_{m_i} \times \prod_{j=1}^f \mathbb{Z}_{k_j}} \]
    \[ [\text{M.C., Lin, 2017}] \]
Digression: F-theory on elliptically fibered Calabi-Yau fourfolds w/specific elliptic fibration (F_{11} polytope) led to D=4 N=1 effective theory

[M.C., Klevers, Peña, Oehlmann, Reuter, ’15]

Standard Model gauge group

\[ SU(3) \times SU(2) \times U(1) \]

\( \mathbb{Z}_6 \)

[M.C., Lin, ’17]

w/ toric bases \( B_3 \) (3D polytopes)

Quadrillion Standard Models (QSMs) with 3-chiral families & gauge coupling unification

[gauge divisors – in class of anti-canonical divisor \( K_c \)]

[Bies, M.C., Donagi,(Liu), Ong, ’21,’22]

Current efforts: determination the exact matter spectra (including vector pair & # of Higgs pairs)

\( \rightarrow \) studies of root bundles on matter curves

c.f., Martin Bies’ talk in parallel session (slides posted)
How to relate these results, encoded in arithmetic structure of elliptic curve – Mordell Weil, to higher-form symmetries?

- Global symmetries, including higher-form ones should be gauged or broken [No Global Symmetry Hypothesis]   
  ...[Harlow, Ooguri ’18]

- In 8D N =1 Supergravity: quantified conditions under which no anomalies due to gauged 1-form symmetry [magnetic version]   
  [M.C., Diriegl, Ling, Zhang, ‘20]

- True for all 8D N=1 string compactifications (beyond F-theory) via (refined) string junction construction   
  [M.C., Diriegl, Ling, Zhang, ‘22]
Fate of higher-form structures in Compact Geometries:

- Decompose $X = \bigcup_n X_n$ into local models $X_n$
  Converse:
  
  Glue $\{X_n\}$ to $X \iff$ Couple $\{\text{SQFT}_n\}$ to resulting one & includes gravity

- Relative Cycles in $X_n$ compactify $\rightarrow$
  (some) defects in $\text{SQFT}_n$ become dynamical - ``gauged''
Fate of higher-form structures in Compact Geometries (continued):

• **Mayer-Vietoris Sequence for covering \( \{X_n\} \):**

\[
\partial_2 : H_2(X) \to \bigoplus_n H_1(\partial X_n)
\]

• **Decomposition of compact two-cycles into a sum of relative cycles associated with each local model**

• **Elliptically fibered geometries:**
  torsional cycles associated w/ Mordell-Weil group decomposition into relative cycles of \( \{X_n\} \)

• **Arguments extend beyond elliptically fibered models:**
  all \( T^4/\Gamma_i \) , some \( T^6/\Gamma_i \),...

[M. C., Heckman, Hübner, Torres, to appear]
Thank you!