Quantum curves and vertex algebras

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Ideas NCBiR

String Math 2022

Quantum curves

Spectral curves are complex curves defined by a polynomial:

$$\Sigma = \{(x, y) \in \mathbb{C}^2 : P(x, y) = 0\}.$$

Quantum curves are differential operators $\widehat{P}$ quantizing polynomial P and annihilating wave function $\widehat{P}\widehat{\psi}(x) = 0$, which (depending on the spectral curve) encode various quantieties:

1. Gromov-Witten invariants in string theory,
2. Jones polynomials in knot theory (conjecturally),
3. Ennumerative geometry invariants,
4. Random matrix correlation functions.

Example: Gaussian unitary ensemble

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$$\widehat{P} = \hbar^2 \partial_x^2 - x\hbar\partial_x + 1$$

annihilate wave function (Hermite polynomial):

$$\widehat{\psi}(x) = \int \det(x - M) e^{-\frac{N}{2}\mathrm{Tr}(M^2)} \mathrm{d}M.$$

Conformal Field Theory

CFT is a quantum field theory invariant under conformal transformations. Locally those transformations are described by Virasoro algebra $\mathcal{V}$, which is spanned by vectors L_n for $n \in \mathbb{Z}$ and a central element c, equipped with a Lie bracket:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m},$$

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Space of states of CFT admits a representation of $\mathcal{V}$.

Verma modules

Definition. Verma module - a representation over $\mathcal{V}$ defined (for any $\Delta, c \in \mathbb{C}$) as:

$$M(\Delta, c) = U(\mathcal{V}) \otimes_{U(\mathcal{V}^{\geq -1})} \mathbb{C}|\Delta\rangle,$$

where U denotes enveloping algebra, $\mathbb{C}|\Delta\rangle$ is 1-dimensional representation of $U(\mathcal{V}^{\geq 0})$ such that $L_0|\Delta\rangle = \Delta|\Delta\rangle$, $c|\Delta\rangle = c|\Delta\rangle$, $L_n|\Delta\rangle = 0$ for $n > 0$.

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Basis of $M(\Delta, c)$ consists of vectors of the form $L_{-i_1} \cdots L_{-i_k}|\Delta\rangle$ for $i_1 > \cdots > i_k > 0$.

Singular vectors

Definition. There exists a special class of vectors in Verma modules $v_{sing} \in M(\Delta, c)$, called singular vectors, not colinear with $|\Delta\rangle$, which satisfy relations

$$L_n v_{sing} = 0$$

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$$\alpha_{r,s} = \frac{r-1}{2}\beta^{-\frac{1}{2}} - \frac{s-1}{2}\beta^{\frac{1}{2}}$$

and $r, s \in \mathbb{N}_{\geq 1}$, $Q = 1 - 6c^2 = \frac{1}{\sqrt{\beta}} - \sqrt{\beta}$, $\beta \in \mathbb{C}$.

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We can write a singular vector in a basis as $A_{r,s}|\Delta_{r,s}\rangle$.

Singular vectors and quantum curves

There is a correspondence between singular vectors and quantum curves, originaly developed from the view of random matrices [1] and later from the view of CFT [2].

[1] *Quantum curves and conformal field theory*, M.Manabe,P.Sułkowski 2015

[2] *From CFT to Ramond super-quantum curves*, L. Hadasz, Z. Jaskólski, M. Manabe, P. Sułkowski,P.C., 2018

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Wave function:

$$\widehat{\psi}_{\alpha,\beta}(x) = \int \Delta(z_1, \ldots, z_N)^{2\beta} \prod_{i=1}^{N} (x - z_i)^{-\frac{\alpha\sqrt{\beta}}{\hbar}} e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{n=1}^{\infty} t_n \sum_{i=1}^{N} z_i^n} \mathrm{d}z_1 \ldots \mathrm{d}z_N,$$

where $\Delta(z_1, \ldots, z_N) = \prod_{i<j}(z_i - z_j)$.

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CFT admits various axiomatic frameworks. We would like to express those results using VOAs.

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Vertex algebras

Definition. Vertex algebra consists of:

- vector space V,
- element $|0\rangle \in V$,
- maps $T : V \to V$ and $Y(v,z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$ for $v \in V$, where $v_{(n)} \in \mathrm{End}(V)$,

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satisfying following conditions:

- $Y(v,z)|0\rangle|_{z=0} = v$
- $T|0\rangle = 0$ and $[T, Y(v,z)] = \partial_z Y(v,z)$
- for any $v_1, v_2 \in V$ there exists $N > 0$ such that $(z-w)^N [Y(v_1,z), Y(v_2,w)] = 0$ (i.e. fields $Y(v_1,z)$ and $Y(v_2,w)$ are local).

Modules

Definition. Vertex operator algebra (VOA): there exists a (conformal) vector $\omega \in V$ such that

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^V z^{-n-2}$$

for some representation of the Virasoro algebra L_n^V on the space V.

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with a linear map $Y_W : V \to \mathrm{End}\, W \otimes \mathbb{C}[[z, z^{-1}]]$ satisfying following conditions:

- $Y_W(|0\rangle, z) = \mathrm{Id}_W$,
- modes of $Y_W(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^W z^{-n-2}$ satisfy commutation relations of Virasoro algebra (with a central charge c^V),
- for any $v \in V$ we have $Y_W(Tv, z) = \partial_z Y_W(v, z)$,
- for any $a, b \in V$ fields $Y_W(a, z)$ and $Y_W(b, w)$ are local.

Example: Fock modules

Definition. Heisenberg algebra: $\mathfrak{h} = \mathrm{lin}\{a_n\}_{n\in\mathbb{Z}} \oplus 1$, Lie bracket:

$$[a_i, a_j] = \frac{1}{2} i\delta_{i+j}, \qquad [a_i, 1] = 0.$$

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$$\mathcal{F}(\alpha) = U(\mathfrak{h}) \otimes_{U(\mathfrak{h}^{\geq 0})} \mathbb{C}|\alpha\rangle,$$

(where $a_n|\alpha\rangle = 0$ for $n > 0$ and $a_0|\alpha\rangle = \alpha|\alpha\rangle$).

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Conformal vector: $\omega = (\frac{1}{2}a_{-1}^2 + Qa_{-2})|0\rangle$, $Q \in \mathbb{C}$. Modes of $Y_{\mathcal{F}(\alpha)}(\omega, z)$:

$$\begin{aligned} L_0^{\mathcal{F}(\alpha)} &= 2\sum_{n=1}^{\infty} a_{-n}a_n + \alpha(\alpha - Q), \\ L_m^{\mathcal{F}(\alpha)} &= \sum_{n\neq 0,m} a_{m-n}a_n + (2\alpha - (m+1)Q)a_m \qquad \text{for } m \neq 0. \end{aligned}$$

Intertwining operators

Let M_1, M_2 and M_3 be modules over certain vertex algebra V and $\mathbb{C}\{x\} = \{\sum_{r\in R} a_r x^r : R \subset \mathbb{C} \text{ countable}\}$.

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such that

1. $I(L_{-1}v, x) = \frac{\partial}{\partial x} I(v, x)$
2. intertwining property:

$$\operatorname*{Res}_{z-w} \left(I(Y_{M_1}(a, z-w)v, w)(z-w)^m \iota_{w,z-w}((z-w)+w)^n \right) =$$
$$\operatorname*{Res}_{z} \left(Y_{M_3}(a, z) I(v, w) \iota_{z,w}(z-w)^m z^n \right) - \operatorname*{Res}_{z} \left(I(v, w) Y_{M_2}(a, z) \iota_{w,z}(z-w)^m z^n \right).$$

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Remark. For $T_3(x)_c = \sum_{n\leq -2} L_n z^{-n-2}$, $T_2(x)_a = \sum_{n>-2} L_n z^{-n-2}$:

$$I(L_{-n}v, x) = \frac{1}{(n-2)!}\left(\partial_x^{n-2} T_3(x)_c I(v, x) + I(v, x)\partial_x^{n-2} T_2(x)_a\right).$$

Intertwining operator - example

Let $M_1 = \mathcal{F}(\alpha)$, $M_2 = \mathcal{F}(\beta)$ and $M_3 = \mathcal{F}(\alpha+\beta)$. Intertwining operators is defined for the highest weight vector:

$$I(|\alpha\rangle, x) = u^{\alpha} \exp\Big(2\alpha \sum_{j=1}^{\infty} \frac{x^j}{j} a_{-j}\Big) \exp\Big(-2\alpha \sum_{j=1}^{\infty} \frac{x^{-j}}{j} a_j\Big) x^{2\alpha a_0},$$

where $u^{\alpha}(a_{-i_1} \cdots a_{-i_k}|\beta\rangle) = a_{-i_1} \cdots a_{-i_k}|\alpha+\beta\rangle$.

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For descendant states:

$$I(a_{-j_1} \dots a_{-j_n}|\alpha\rangle, x) =: D^{j_1-1}\phi(x) \cdots D^{j_n-1}\phi(x) I(|\alpha\rangle, x) :,$$

where $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ and $D^n = \frac{1}{n!}\frac{\partial^n}{\partial x^n}$.

Screening operator

Definition. Screening operator is a $\mathcal{V}$-homomorphism of Fock modules, defined by an integral:

$$\Sigma_{N,\beta,b} = \int_\Gamma K^\beta(z_1, \ldots, z_N) \prod_{i=1}^{N} z_i^{-b-1} \mathrm{d}z_1 \ldots \mathrm{d}z_N,$$

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and $\Gamma \in H_N(M_N,\Psi_\beta)$ is a twisted cycle

$(M_N = \{(z_1,\ldots,z_N) \in (\mathbb{C}^*)^N : z_i \neq z_j \quad dla \quad i \neq j\}$, $\Psi_\beta = \prod_{i\neq j}\left(1-\frac{z_i}{z_j}\right)^\beta$).

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Remark. Screening operators maps singular vectors to either highest weight vectors, other singular vectors or to 0.

Wave function

Define an operator $\Phi_\hbar$, which maps a_{-i} to $\frac{it_i}{2\hbar}$, and operators a_i to $\hbar\frac{\partial}{\partial t_i}$ and let $b = (\alpha - Q)\sqrt{\beta} - N\beta$, $N \in \mathbb{Z}_{>0}$, $\beta \in \mathbb{C}$ and $b \in \mathbb{Z}$.

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Definition. Wave function $\widehat{\psi}_\alpha(v, x) \in \mathbb{C}[[t_0, t_1, \dots]] \otimes \mathbb{C}[\hbar^{\pm 1}] \otimes \mathbb{C}\{x\}$ is defined by an equation:

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Remark. We can represent $\widehat{\psi}_\alpha(|\alpha\rangle, x)$ as an eigenvalue integral

$$\widehat{\psi}_\alpha(|\alpha\rangle, x) = \exp\left(-\frac{\alpha}{\hbar}\sum_{n=1}^{\infty} t_n x^n\right) \int_\Gamma \Delta(z_1, \dots, z_N)^{2\beta} \prod_{i=1}^{N} (z_i - x)^{-2\alpha\sqrt{\beta}} \cdot \prod_{i=1}^{N} \exp\left(-\frac{\sqrt{\beta}}{\hbar}\sum_{n=1}^{\infty} t_n z_i^n\right) \prod_{i=1}^{N} z_i^{-b-1} \mathrm{d}z_1 \dots \mathrm{d}z_N.$$

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Dyson index β for specific values corresponds to appropriate ensembles of random matrices: $\beta = 1$ for GOE, $\beta = 2$ for GUE, and $\beta = 4$ for GSE.

Representation

Theorem.[3] *Assume that* $N \in \mathbb{Z}_{>0}$, $\beta \in \mathbb{C}$ *and* $b \in \mathbb{Z}$. *Then there exists a representation* $\widehat{L}_{-n}$ *of the negative modes of the Virasoro algebra*

$$\widehat{L}_{-n}\widehat{\psi}_{\alpha}(v, x) = \widehat{\psi}_{\alpha}(L_{-n}v, x),$$

[3] *On quantum curves, Airy structures and supersymmetry*, P.C., 2020

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$$\widehat{L}_{-n}\widehat{\psi}_{\alpha}(v,x) = \widehat{\psi}_{\alpha}(L_{-n}v,x),$$

such that:
$\widehat{L}_{-1} = \partial_x$, *and for* $n \geq 2$*:*

$$\begin{aligned}\widehat{L}_{-n} &= \frac{1}{\hbar^2 (n-2)!}\partial_x^{n-2}\Big(\frac{1}{4}(\partial_x W(x))^2 + \frac{Q\hbar}{2}\partial_x^2 W(x) \\ &+ \hbar^2 \sum_{k=0}^{\infty} x^k \sum_{m=k+2}^{\infty} m t_m \partial_{t_{m-k-2}} + (\alpha - N\sqrt{\beta})\hbar \sum_{m=0}^{\infty}(m+2)t_{m+2}x^m\Big),\end{aligned}$$

where $W(x) = \sum_{n=1}^{\infty} t_n x^n$.

[3] *On quantum curves, Airy structures and supersymmetry*, P.C., 2020

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Corollary: for $a, b, k > 0$ i $k \geq a+b+2$:

$$\sum_{n=b}^{k-2-a}\binom{k-n-2}{a}\binom{n}{b}(2n-k+2) = (b-a)\binom{k}{a+b+2}.$$

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Quantum curves

Theorem. *Let* $\Delta, Q, \alpha_{r,s}$ *as before,* $\beta > 0$, $\alpha = Q - \alpha_{r,s}$, $2N = s - 1$, $2b = 1 - r$.
If $v_s = A_{r,s}|\Delta\rangle$ *is a singular vector, then:*

$$\widehat{A}_{r,s}\widehat{\psi}_{\alpha}(|\alpha\rangle, x) = 0$$

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Example:

$$\widehat{A}_{1,3} \quad = \quad \partial_x^3 - 4\beta^{-1}\widehat{L}_{-2}\partial_x + (-2\beta^{-1} + 4\beta^{-2})\widehat{L}_{-3},$$

where we set $r = 1$, $s = 3$ so $b = 0$ and $N = 1$.

W-algebras

$\mathcal{W}$-algebras are vertex algebras generated by a finite number of fields[4]. Algebras of its modes are not necessarily Lie algebras, as Lie bracket of two operators in general lies in the enveloping algebra.

[4] *Higher Airy structures, W algebras and topological recursion*, G. Borot, V. Bouchard, N. K. Chidambaram, T. Creutzig, D. Noshchenko, 2018

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Let $w \in \mathcal{F}(\alpha)$ be a vector in the Fock module, $Y(w,z) = \sum_k W_k z^{-k-1}$ be the corresponding field and I be an intertwining operator. Then

$$I(W_{-k}v, z)|0\rangle = \frac{1}{(k-1)!}\partial^{k-1}\Big(\sum_{k<0} W_k z^{-k-1}\Big) I(v,z)|0\rangle$$

for any $v \in \mathcal{F}(\alpha)$.

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Open question: Can we get quantum curves for $\mathcal{W}$-algebras ?

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Thank you for your attention.