Plumbing Graphs with Matter

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based on a work (to appear) with P. Sułkowski

Motivation

To understand

- mixed Chern-Simons levels of 3d $\mathcal{N} = 2$ theories [see e.g.Dorey,Tong,1999]
- construction of theories using the 3d-3d correspondence [Terashima, Yamazaki, 2011][Dimofte, Gaiotto, Gukov, 2011]
- knots-quivers correspondence [Kucharski, Reineke, Stosic, Sułkowski, 2017]

3d $\mathcal{N} = 2$ Abelian theories

- Gauge groups: $U(1) \times \cdots \times U(1)$.
- Mixed Chern-Simons levels

$$S_{CS} = k_{ij} \int A_i dA_j \,, \quad k_{ij} = k_{ji} \,.$$

- The effective Chern-Simons levels $k_{ij}^{\text{eff}} \in \mathbb{Z}$.
- Vector multiplet: FI parameter ξ
- Chiral multiplet: mass parameter $m \to$ flavor symmetry $U(1)_m$.
- Superpotential: $\mathcal{W} = \Phi_1 \Phi_2 \Phi_3 \cdots$.

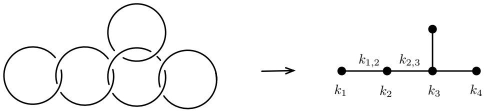
3d-3d correspondence

- $6d$ $(2,0)$ theories $\xrightarrow{M_3}$ $3d$ $N=2$ theories
- Wrap one M5-brane $\xrightarrow{M_3}$ Abelian theories
- M_3 can be plumbing manifolds.

  [Gadde,Gukov,Putrov,2013][Gukov,Putrov,Vafa,2016][Gukov,Pei,Putrov,Vafa,2017]

Plumbing graphs

- Plumbing manifolds are represented by plumbing graphs:



- Linking numbers K_{ij} between circles:

$$K_{ij} = \begin{cases} k_i\,, & i = j \\ k_{ij}\,, & i \neq j \end{cases}\,.$$

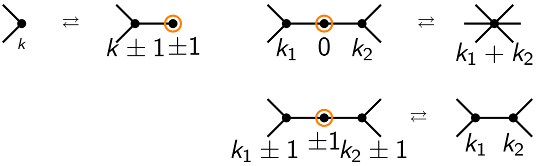
- Engineer Abelian gauge theories:

$$U(1) \times \cdots \times U(1), \text{ with mixed CS levels } K_{ij}\ .$$

[Gadde,Gukov,Putrov,2013]

Kirby moves for gauge nodes

- Kirby moves produce equivalent graphs



- Kirby moves = integrating out gauge nodes

The contribution of $U(1)_k$ can be separated:

$$Z_{S_b^3} = \int dx\ e^{kx^2+(\cdots)x} \times (\text{other terms})\,.$$

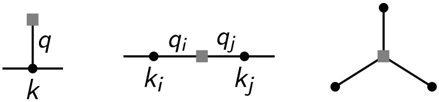
[Gadde,Gukov,Putrov,2013]

Problem

- How to describe matter? such as fundamental matter **F**, anti-fundamental matter **AF**, and so on.
- Are there similar Kirby moves for matter?

Matter

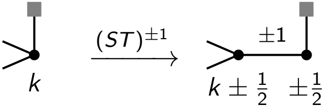
- We conjecture it is possible to add matter.
- A new notation: we denote matter by ■.



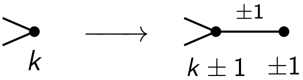
- q_i are charges under gauge nodes.

Kirby move for matter: ST-transformation

- The Lagrangians of 3d theories enjoy a $SL(2,\mathbb{Z})$ action, where $ST \in SL(2,\mathbb{Z})$ with $(ST)^3 = 1$. [Witten, 2003][Dimofte, Gaiotto, Gukov, 2011]
- ST introduces a new gauge node and mixed CS levels.
- Graphically,

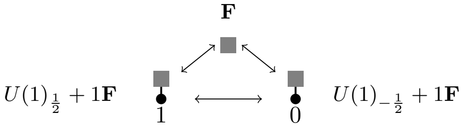


- It is analogous to



Kirby move for matter: ST-transformation

- ST leads to the triality: [Dimofte, Gaiotto, Gukov, 2011]



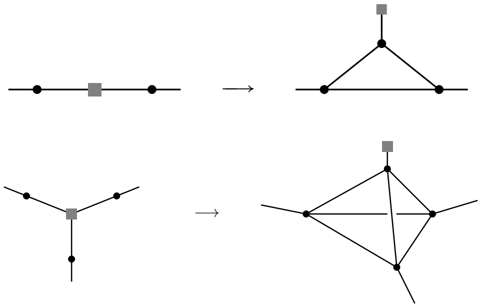
- Their sphere partition functions are equivalent

$$(ST)^{\pm} \ : \quad s_b\left(\frac{iQ}{2} - z\right) = e^{\mp \frac{\pi i}{2}\left(\frac{iQ}{2} - z\right)^2} \int dx\, e^{\mp \frac{\pi i}{2} x^2 \mp 2\pi i\, zx - \frac{\pi Q}{2} x} s_b\left(\frac{iQ}{2} - x\right) .$$

- This identity can be used as a nontrivial replacement to transform generic matter.

Kirby moves for matter: ST-transformation

- A generic matter can be turned into a fundamental matter **F**:



Generic Kirby moves

Kirby moves $+$ ST-transformations

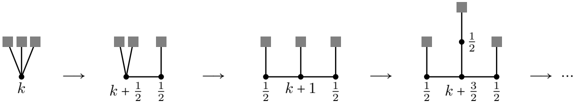
to generate equivalent plumbing graphs.

Equivalent graphs $\longleftrightarrow$ Equivalent theories

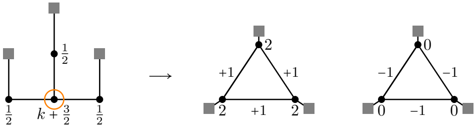
Generic Kirby moves lead to dual theories.

Examples: $U(1)_k + 3\mathbf{F}$

- Kirby moves $\rightarrow$ many graphs



- For $k = -\frac{5}{2}$ or $-\frac{1}{2}$, symmetric graphs

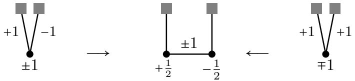


- For $k = \pm\frac{3}{2}$, bifundamentals

Examples: mirror theories

- Mirror dual theories have equivalent plumbing graphs.
- An example is

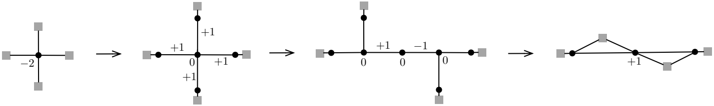
$$U(1)_{\pm 1} + 1\mathbf{F} + 1\mathbf{AF} \xleftrightarrow{\text{mirror}} U(1)_{\mp 1} + 2\mathbf{F}$$



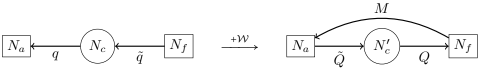
- The Abelian mirror pair [Dorey, Tong, 1999]

$$U(1)_{-\frac{N_f}{2}} + N_f\mathbf{F} \xleftrightarrow{\text{mirror}} [1] - U(1) - U(1) - \cdots - U(1) - [1]$$

is related through Kirby moves. For example,



Seiberg duality

- Could plumbing graphs encode superpotentials?
- 

- Seiberg duality $\longrightarrow$ an additional matter + a superpotential
- We consider the basic mirror dual pair

$$1\mathbf{F} + 1\mathbf{AF} \quad \longleftrightarrow \quad U(1)_0 + 1\mathbf{F} + 1\mathbf{AF} + 1\mathbf{Adj} \ \text{ with } \mathcal{W} = Q\tilde{Q}\Phi_{adj} \, .$$

- Their sphere partition functions are equivalent

[Dimofte, Gaiotto, Gukov, 2011]

$$Z_{S_b^3}^{LHS} = Z_{S_b^3}^{RHS} : \ s_b(y \pm z) = s_b\left(iQ/2 - 2y\right) \int dx \, e^{-2\pi i \, zx} s_b\left(iQ/2 \pm x - y\right) \, .$$

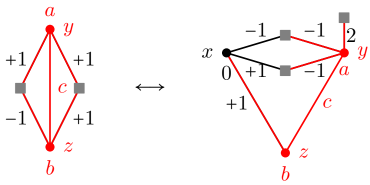
- Flavor symmetry $U(1)_y \times U(1)_z$.

Seiberg-like duality

- To generate interesting graphs, we should gauge flavor symmetries.
- After gauging, the dual pair becomes

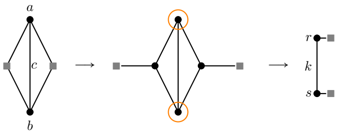
$$\boxed{\int dy\,dz\,[\ldots]\;Z^{LHS}_{S^3_b} = \int dy\,dz\,[\ldots]\;Z^{RHS}_{S^3_b}}$$

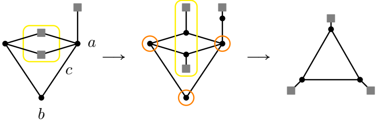
$$[\ldots] := e^{-\pi i (ay^2+bz^2+2c\,yz)}\,e^{2\pi i\,(\xi_y y+\xi_z z)}\,.$$



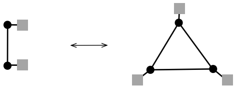
Seiberg-like duality

- Kirby moves simplify Seiberg-like dual graphs



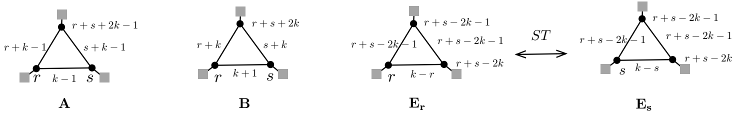


- The Seiberg-like duality becomes a 2-3 move



Triangles A,B,E

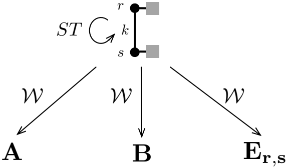
- Triangles encoding superpotentials are not unique:



- Cases **A** and **B** are firstly found in [Ekholm, Kucharski, Longhi, 2019], which are named unlinking and linking. These triangles are very useful to study knot invariants [see Larraguível and Noshchenko's poster].

Triangles A,B,E

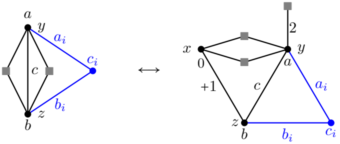
- The relation between these triangles is given by



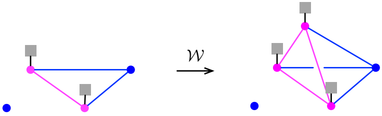
- Triangles **A**,**B** and **E** are related by $\mathcal{W}^{-1} \circ (ST)^* \circ \mathcal{W}$.

Local duality

- Seiberg-like duality is a local duality:

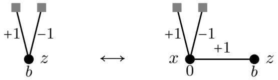


- Kirby moves simplify this dual pair:

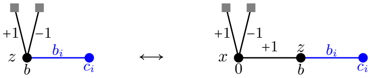


Partially gauging

- Gauging only the flavor symmetry $U(1)_z$ decouples a matter. Then we have a 2-2 move



- If this dual pair couples to other gauge nodes



- These dual graphs also satisfy cases **A**, **B** and **E**.

Higgsing

- If one adjusts the mass parameters for a **F** and an **AF** properly, then

$$1\mathbf{F} + 1\mathbf{AF} \to 1\,.$$

Graphically,

- This is the Higging of a D5-brane in the brane web description.

Outlook

- Relation to higher form symmetry.[Eckard,Kim,Schafer-Nameki,Willett,2019]
- Construct nonabelian theories and more generic dualities.
- Introduce matter to plumbing graphs using M5-branes.
- Relation to other constructions: CY4, orbifolds, BPS quivers, and brane webs.

Thank you!