q-series for $SO(3)$ and $OSp(1|2)$

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String-Math 2022,
13 July 2022

Outline

1. Background
   (a) *WRT* invariant
   (b) $WRT_{SU(2)}$ invariant for plumbed manifold M_3
      (i) Plumbed 3-manifold M_3
      (ii) WRT invariant
      (iii) Kirby-Neumann moves
   (c) Recent development in categorification of *WRT*
2. *WRT* for $SO(3)$ and $OSp(1|2)$
   (a) $SO(3)$ WRT invariant
   (b) $OSp(1|2)$ WRT invariant
3. q-series
   (a) For $SO(3)$ WRT invariant
   (b) For $OSp(1|2)$ WRT invariant
4. Conclusion

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Witten-Reshetikhin-Turaev invariant(WRT)

[Witten (1988)], [Reshetikhin-Turaev (1990)]

Consider a Chern-Simons theory whose action is a 3-form on a three-manifold M_3:

$$S_{CS}[A] = \frac{k}{4\pi} \int_{M_3} \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

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$$Z[M] = \int \mathcal{D}A exp[iS_{CS}]$$

proven to be a 3-manifold invariant with $k \in \mathbb{Z}$, which is also known as WRT invariant.

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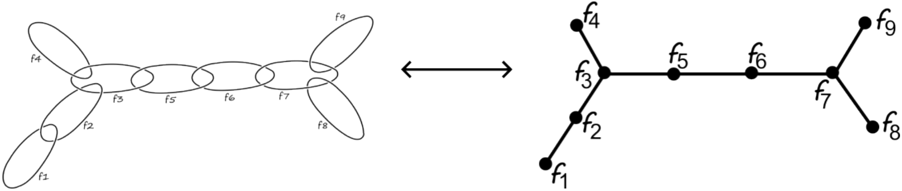
Plumbed 3-manifold M_3

Plumbed 3-manifolds $M_3(\Gamma)$ is a very large class of 3-manifolds associated to a plumbing graph Γ.



$WRT_{SU(2)}$ invariant for plumbed manifold M_3

One associate a linking matrix M to a graph Γ as follows:

$$M_{v_1,v_2} = \begin{cases} 1, & v_1, v_2 \text{ connected,} \\ f_v, & v_1 = v_2 = v, \\ 0, & \text{otherwise.} \end{cases} \qquad v_i \in \text{Vertices of } \Gamma \cong \{1, \dots, L\}$$

$H_1(M_3, \mathbb{Z}) \cong \text{Coker } M = \mathbb{Z}^L / M\mathbb{Z}^L$

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$$Z_{SU(2)_{k+2}}\left[M_3(\Gamma)\right] = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right) \tau_{k+2}\left[M_3(\Gamma)\right]$$

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$$\tau_{k+2}[M_3(\Gamma)] = \frac{F[\mathcal{L}(\Gamma)]}{F[\mathcal{L}(+1)]^{b_+} F[\mathcal{L}(-1)]^{b_-}} \tag{1}$$

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the colored link invariant $V[\mathcal{L}(\Gamma)]_{n_1,n_2,\ldots,n_L}$ is the colored Jones polynomial for $SU(2)$ with $\mathbb{q} = Exp\left[\frac{2\pi i}{k+2}\right]$:

$WRT_{SU(2)}$ invariant for plumbed manifold M_3

$$V[\mathcal{L}(\Gamma)]_{n_1,n_2,\ldots,n_L} = \frac{2i}{\mathbb{q}^{1/2} - \mathbb{q}^{-1/2}} \prod_{v \in \text{Vertices} \cong \{1,\ldots,L\}} \mathbb{q}^{\frac{f_v(n_v^2-1)}{4}} \times$$

$$\left(\frac{2i}{\mathbb{q}^{\frac{n_v}{2}} - \mathbb{q}^{-\frac{n_v}{2}}}\right)^{\deg(v)-1} \prod_{(v_1,v_2) \in \text{Edges}} \left(\frac{\mathbb{q}^{\frac{n_{v_1} n_{v_2}}{2}} - \mathbb{q}^{-\frac{n_{v_1} n_{v_2}}{2}}}{2i}\right)$$

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this can be written in the form of modular S and T matrices:

$$V[\mathcal{L}(\Gamma)]_{n_1,n_2,\ldots,n_L} = \frac{1}{S_{00}} \prod_{i=1}^{L} \left(T_{(n_i-1),(n_i-1)} \right)^{f_i} \left(S_{0,(n_i-1)} \right)^{1-\deg(v_i)} \times \prod_{\text{Edges} \in (v_i,v_j)} S_{(n_i-1),(n_j-1)} \quad (2)$$

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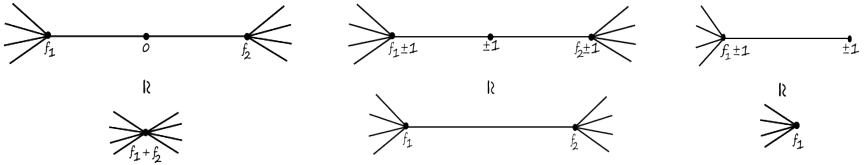

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WRT invariant $\tau_k[M_3(\Gamma)]$ can be written as follows:

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$$\hat{Z}_b(q) = q^{-\frac{3L + \sum_v f_v}{4}} \cdot \text{v.p.} \int_{|z_v|=1} \prod_{v \in \text{Vertices}} \frac{dz_v}{2\pi i z_v} \left(z_v - 1/z_v\right)^{2-\deg(v)} \cdot \Theta_b^{-M}(z)$$

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Examples

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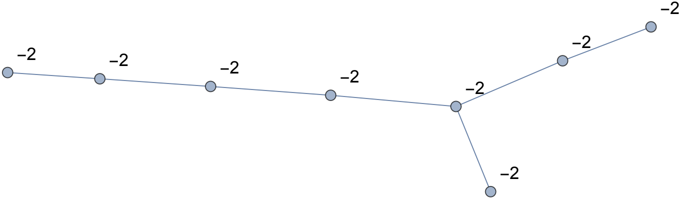

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Examples

$$H_1(M_3) = 0$$

$$\hat{Z}_0 = q^{-3/2}(1 - q - q^3 - q^7 + q^8 + q^{14} + q^{20} + q^{29} - q^{31} - q^{42} + \cdots)$$

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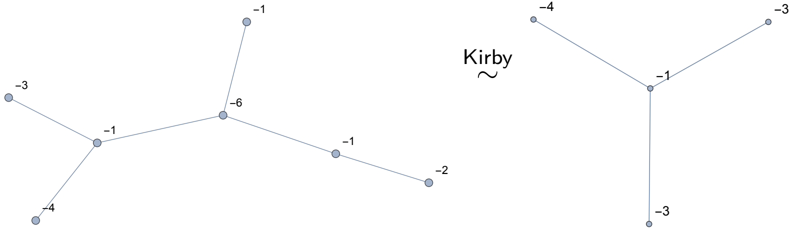

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Examples

$$H_1(M_3) = \mathbb{Z}_3$$

$$\hat{Z} = \begin{pmatrix} 1 - q + q^6 - q^{11} + q^{13} - q^{20} + q^{35} + O\left(q^{41}\right) \\ q^{5/3}\left(-1 + q^3 - q^{21} + q^{30} + O\left(q^{41}\right)\right) \end{pmatrix}$$

WRT for $SO(3)$ and $OSp(1|2)$

$SO(3)$ WRT invariant

[Kirby, Melvin (1991)], [Kaul, Ramadevi (2000)], [Beliakova, Le (2006)], [Beliakova, Blanchet, Le (2007)]

On plugging the expression of S and T matrices for $SO(3)$ in equation(2), we find the colored link invariant:

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where $\mathfrak{q} = Exp\left[\frac{\pi i}{k+1}\right]$ and $k \in 2\mathbb{Z}$.

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$SO(3)$ WRT invariant

The quantity $F[\mathcal{L}(\Gamma)]$ is given by

$$F[\mathcal{L}(\Gamma)] \equiv \sum_{m \in \{1,3,\ldots,2k+1\}^L} V[\mathcal{L}(\Gamma)]^{SO(3)}_{m_1,\ldots,m_L} \prod_{v=1}^{L} \frac{\mathbb{q}^{m_v/2} - \mathbb{q}^{-m_v/2}}{\mathbb{q}^{1/2} - \mathbb{q}^{-1/2}}$$

On plugging this in equation(1), we get the WRT invariant.

WRT for $SO(3)$ and $OSp(1|2)$

[Ennes, Ramadevi, Ramallo, Sanchez de Santos (1997)], [Chauhan, Ramadevi (20XX)]

$OSp(1|2)$ WRT invariant

Again substituting the modular matrices for $OSp(1|2)$, we find the following colored link invariant,

$$V[\mathcal{L}(\Gamma)]^{OSp(1|2)}_{m_1,\ldots,m_L} = \frac{2i}{\mathbb{q}^{1/2}+\mathbb{q}^{-1/2}} \prod_{v\,\in\,\text{Vertices}\,\cong\,\{1,\ldots,L\}} \mathbb{q}^{\frac{f_v(m_v^2-1)}{4}} \left(\frac{2i}{\mathbb{q}^{m_v/2}+\mathbb{q}^{-m_v/2}}\right)^{\deg(v)-1} \prod_{(v_1,v_2)\,\in\,\text{Edges}} \frac{\mathbb{q}^{m_{v_1}m_{v_2}/2}+\mathbb{q}^{-m_{v_1}m_{v_2}/2}}{2i}$$

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q-series invariant

[Chauhan, Ramadevi (20XX)]

$SO(3)$ q-series

The $SO(3)$ WRT invariant is decomposed as follows

$$\tau_{k=2j}[M_3(\Gamma)] = \frac{1}{2\left(\mathbb{q}^{1/2} - \mathbb{q}^{-1/2}\right)|\det M|^{1/2}} \sum_{a \in \mathrm{Coker}\, M} e^{-\pi i (2j+1)(a, M^{-1} a)} \sum_{b \in 2\mathrm{Coker}\, M + \delta} e^{-\pi i \left(a, M^{-1}(b \; + MI \;)\right)} \lim_{q \to \mathbb{q}} \hat{Z}_b(q).$$

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Surprisingly, we found out that $SO(3)$ q-series is same as that of $SU(2)$ q-series.

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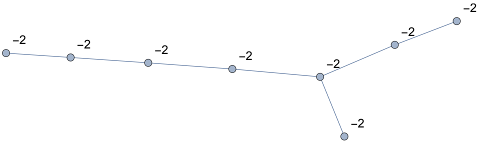

q-series invariant

[Chauhan, Ramadevi (20XX)]

Examples:

$$H_1(M_3) = 0$$

$$\hat{Z}_0 = q^{-3/2}(1 + q + q^3 + q^7 + q^8 + q^{14} + q^{20} - q^{29} + q^{31} - q^{42} - q^{52} + \cdots)$$

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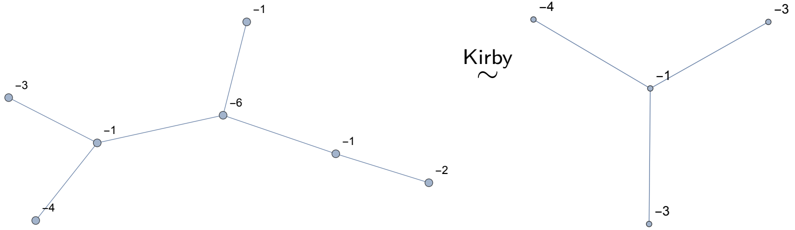

q-series invariant

[Chauhan, Ramadevi (20XX)]

Examples:

$$H_1(M_3) = \mathbb{Z}_3$$

$$\hat{Z} = \begin{pmatrix} 1 + q + q^6 + q^{11} - q^{13} - q^{20} - q^{35} + O\left(q^{41}\right) \\ q^{5/3}\left(1 + q^3 - q^{21} - q^{30} + O\left(q^{41}\right)\right) \end{pmatrix}$$

Some open problems and future directions

- higher rank gauge groups $SO(N)$ and $OSp(n|2n)$ with n being odd
- check whether the equality of q-series for $SU(2)$ and $SO(3)$ holds in general
- more complicated 3-manifolds
- refinements

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Thank you for your attention