Categorical Enumerative Invariants

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History, part 1: 1990's

- Candellas-de la Ossa-Green-Parkes '92: Enumerative Mirror Symmetry (EMS)

  Computed the sequence

  $$2875, 609250, 317206375, 242467530000, 229305888887625, \ldots$$

  of Gromov-Witten invariants of genus 0, degree d of a quintic Calabi-Yau threefold X.

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  of Gromov-Witten invariants of genus 0, degree d of a quintic Calabi-Yau threefold X.

- This computation was done by identifying these numbers with the coefficients of a solution of the Picard-Fuchs equation which governs the variation of Hodge structures on the family of mirror quintics $\check{X}_\psi$.

History, part 1: 1990's (cont'd)

- Kontsevich 1994: Homological Mirror Symmetry (HMS)
  There exists a mirror map $\psi \leftrightarrow \rho$ such that

  $$\mathbf{D}^b_{\mathrm{coh}}(\check{X}_\psi) \cong \mathrm{Fuk}(X, \rho).$$

  Here ρ is a parameter in the Novikov ring, roughly parametrizing the volume of the symplectic form.

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- Kontsevich's (implied) conjecture:

  $$\text{Homological MS} \Rightarrow \text{Enumerative MS}$$

Intuitive idea

There should exist an invariant $F_0(\mathscr{C})$ (a complex number) associated to a Calabi-Yau category $\mathscr{C}$ (over the complex numbers).

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If we compute it for $\mathscr{C} = \mathbf{D}^b_{\mathsf{coh}}(\check{X}_\psi)$ we get a function $F_0(\psi)$, which should be the COGP solution of Picard-Fuchs.

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HMS then implies that under the mirror map $\psi \leftrightarrow \rho$ we will have

$$F_0(\mathbf{D}^b_{\mathrm{coh}}(\check{X}_\psi)) = F_0(\mathrm{Fuk}(X, \rho))$$

so the two functions are the same (main statement of EMS).

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- (Insertions) Gromov-Witten theory more generally defines gravitational descendants

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- To make sense of mirror symmetry one needs to choose a large complex structure limit point. Where does this choice enter the categorical theory?

History, part 2: 2000's

- (Kontsevich-Soibelman, Costello, ∼2000): Explicit construction

  $$\{\text{Smooth cyclic } A_\infty \text{ algebras}\} \leftrightarrow \{\text{2d extended TFTs}\}$$

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- (Costello 2005): Non-explicit construction of

$$\langle \tau_{i_1}(\alpha_1), \ldots, \tau_{i_n}(\alpha_n) \rangle_{g,n}^{A,s}$$

  for any smooth Calabi-Yau A_∞-algebra A, splitting s of the Hodge filtration of A, and $\alpha_1, \ldots, \alpha_n \in HH_*(A)$.

  (Construction goes through the extended 2d TFT of A.)

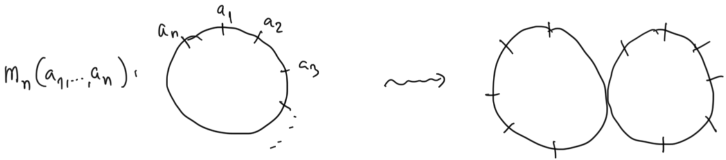
Cyclic A_∞ algebras

A_∞-algebras are the homotopy generalization of ordinary algebras, where we relax the associativity constraint.

A cyclic A_∞ algebra A consists of:

- a graded vector space A;
- operations $m_k : A^{\otimes k} \to A[2-k]$ satisfying generalized associativity axioms;
- a non-degenerate pairing $\langle -, - \rangle : A^{\otimes 2} \to \mathbb{C}$ compatible with multiplications (analogue of pairing of Frobenius algebra.)

Graphically we think of cyclic operations as results of disk insertions, compatible with disk bubbling.



Cyclic A_∞ algebras (cont'd)

How do cyclic A_∞ algebras arise?

- Fukaya categories are naturally presented as cyclic A_∞.
- For Calabi-Yau variety X and $\mathscr{C} = \mathbf{D}^b_{\text{coh}}(X)$ we need to compute $A = \text{End}(E)$ for any generator E.

  Done explicitly for X = elliptic curve (Polishchuk).

  Recursive procedure for X = hypersurface in $\mathbf{P}^n$ (Tu).
- For categories of matrix factorizations we can take the generator to be k^{stab}, the stabilization of the residue field. Resulting A_∞ algebra computed by Dyckerhoff (special cases), Tu (homogeneous potential).

New actor: the splitting s

Costello's invariants require a new ingredient, a *splitting of the Hodge filtration*.

Example: If $\check{X}$ is an elliptic curve, $\mathscr{C} = \mathbf{D}^b_{\mathrm{coh}}(\check{X})$, have a natural short exact sequence

$$0 \to H^0(\check{X}, \Omega^1_{\check{X}}) \to H^1_{\mathrm{dR}}(\check{X}) \to H^1(\check{X}, \mathscr{O}_{\check{X}}) \to 0.$$

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(This is categorical: the Hochschild to cyclic spectral sequence

$${}^1E = HC_*(\mathscr{C}) \Rightarrow HH_*(\mathscr{C})[[u]]$$

degenerates at ${}^1E$ for any $\mathbb{Z}$-graded A_∞ algebra – Kaledin.)

Splittings (cont'd)

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Similar to Witten's point of view: for the mirror quintic $\check{X}$ we need to write the symplectic space $H^3_{\mathrm{dR}}(\check{X})$ as a sum of two Lagrangians $L_1 \oplus L_2$, where $L_1 = H^{3,0} \oplus H^{2,1}$, $L_2 = H^{1,2} \oplus H^{0,3}$.

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In order to compute categorical enumerative invariants we need to choose this splitting in the whole family $\check{X}_\psi$, or at least near a large complex structure limit.

Choices of splittings

1. *(Complex conjugate splitting)*: Use $L_2 = \overline{L_1}$. It is modular (i.e., defined everywhere), but does not vary holomorphically with ψ, so $F_g(\mathbf{D}^b_{\mathrm{coh}}(\check{X}_\psi))$ will not be a holomorphic function of ψ.

For elliptic curve $\check{X}$ this corresponds to the familiar decomposition

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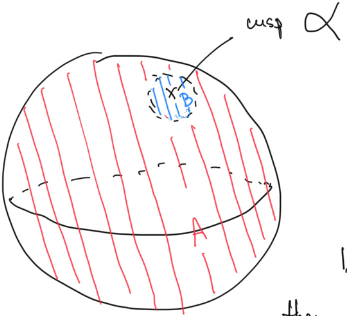
2. *(Monodromy invariant spitting)*: use monodromy around any of the special points of the moduli space (large complex structure points, orbifold points, ...) to fix splitting. Varies holomorphically, not modular.

For elliptic curve with modular parameter τ, around large complex structure point, this corresponds to the decomposition

$$H^1_{\text{dR}}(\check{X}) = \langle dz, \frac{dz - d\bar{z}}{\tau - \bar{\tau}} \rangle.$$

Why do we need the splitting?

Consider the $g = 1$, $n = 1$ invariant: $\overline{M}_{1,1} = \mathbf{P}^1 = S^2$.



$$\langle \alpha \rangle_{1,1} = \int_{\overline{M}_{1,1}} \Omega_{1,1}(\alpha) = \underbrace{\int_A \Omega_{1,1}(\alpha)}_{\text{computed from extended TFT}} + \underbrace{\int_B \Omega_{1,1}(\alpha)}_{\text{not computable from TFT.}}$$

If we knew $\Omega_{1,1}(\alpha)\big|_B = d\,\Omega'_{1,1}(\alpha)$

then

$$\int_B \Omega_{1,1}(\alpha) = \int_{\partial B} \Omega'_{1,1}(\alpha)$$

which can be computed from TFT.

The choice of splitting is determining the form $\Omega'_{1,1}(\alpha)$.

CEI = GW conjecture

Conjecture (Costello, 2005). For $\mathscr{C} = \mathrm{Fuk}(X, \rho)$, $s =$ monodromy invariant splitting around large volume limit point, Costello's invariants compute the Gromov-Witten potential of X.

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Theorem (C, Costello, Tu, 2020). There exist combinatorial, explicit formulas for Costello's invariants. Given a cyclic A_∞ algebra A and a splitting s of its Hodge filtration there exists an explicit algorithm to compute

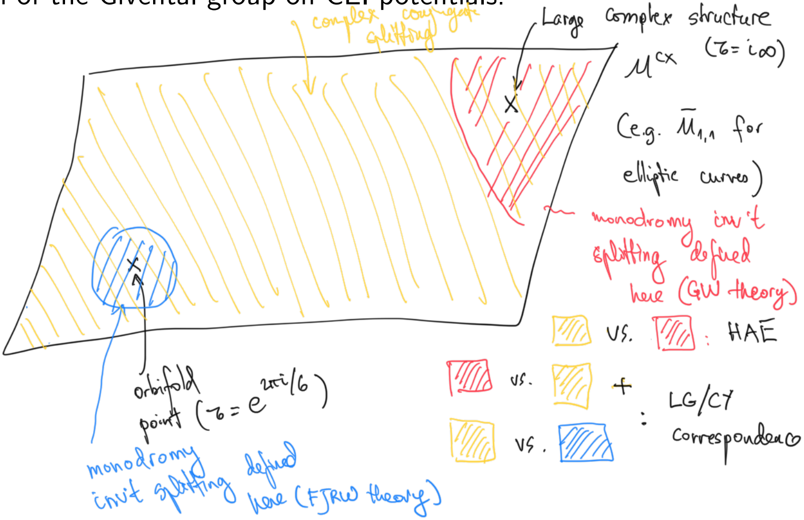
$$\langle \tau_{i_1}(\alpha_1), \ldots, \tau_{i_n}(\alpha_n) \rangle_{g,n}^{A,s}.$$

(Very computationally intensive, almost useless in practice.)

Can now verify Costello's conjecture for $X = T^2$ (2-torus) and $g \leq 2$ (soon expected for $g \leq 5$).

The power of using different splittings

All splittings form a torsor over the *Givental group*. We have formulas (in terms of summations over stable graphs) for the action of the Givental group on CEI potentials.



Other computations, future directions

Known results:

- A_n singularities in $g = 0$, $g = 1$: $\mathscr{C} = \mathrm{MF}(\mathbb{A}^1, x^{n+1})$. Interesting choice of splitting, no monodromy choice.
- String, divisor axioms from GW theory hold for CEIs (work in progress, Shen-Tu).

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Problems of interest:

- Can we distinguish two non-derived equivalent CY threefolds studied by Aspinwall-Morrison using $g = 1$ CEIs?
- Computations of CEIs for Kuznetsov components of cubic fourfolds? Relationship to rationality?