Towards F-theory MSSMs

Martin Bies

University of Pennsylvania

String Math Conference, Warsaw – July 13, 2022

With M. Cvetič, R. Donagi, M. Liu, M. Ong – 2102.10115, 2104.08297, 2205.00008

Overview of this talk: Goal, Challenge and Tool

Motivation

- Go beyond chiral spectrum of String theory standard model constructions.
  $\Rightarrow$ For MSSM, need one **massless vector-like pair** to accommodate the **Higgs**.

Overview of this talk: Goal, Challenge and Tool

Motivation

- Go beyond chiral spectrum of String theory standard model constructions.
  $\Rightarrow$ For MSSM, need one **massless vector-like pair** to accommodate the **Higgs**.
- F-theory is cool. [Review in two slides.]
  $\Rightarrow$ **Quadrillion F-theory standard models (QSMs)**. [Cvetič Halverson Lin Liu Tian '19]

Overview of this talk: Goal, Challenge and Tool

Motivation

- Go beyond chiral spectrum of String theory standard model constructions.
  $\Rightarrow$ For MSSM, need one **massless vector-like pair** to accommodate the **Higgs**.
- F-theory is cool. [Review in two slides.]
  $\Rightarrow$ **Quadrillion F-theory standard models (QSMs)**. [Cvetič Halverson Lin Liu Tian '19]

Goal of this talk:

Compute vector-like spectra in reps. $(\overline{\mathbf{3}}, \mathbf{2})_{1/6}$, $(\overline{\mathbf{3}}, \mathbf{1})_{-2/3}$, $(\mathbf{1}, \mathbf{1})_1$ of F-theory QSMs.

(Sadly, $(\overline{\mathbf{1}}, \mathbf{2})_{-1/2}$ is currently too hard for our techniques. We hope to get there in the future.)

Overview of this talk: Goal, Challenge and Tool

Motivation

- Go beyond chiral spectrum of String theory standard model constructions.
  $\Rightarrow$ For MSSM, need one **massless vector-like pair** to accommodate the **Higgs**.
- F-theory is cool. [Review in two slides.]
  $\Rightarrow$ **Quadrillion F-theory standard models (QSMs)**. [Cvetič Halverson Lin Liu Tian '19]

Goal of this talk:

Compute vector-like spectra in reps. $(\overline{\mathbf{3}}, \mathbf{2})_{1/6}$, $(\overline{\mathbf{3}}, \mathbf{1})_{-2/3}$, $(\mathbf{1}, \mathbf{1})_1$ of F-theory QSMs.

(Sadly, $(\overline{\mathbf{1}}, \mathbf{2})_{-1/2}$ is currently too hard for our techniques. We hope to get there in the future.)

Challenge

In global F-theory compactifications, vector-like spectra are **non-topological**.

[M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. Mayrhofer Weigand '18]

Overview of this talk: Goal, Challenge and Tool

Motivation

- Go beyond chiral spectrum of String theory standard model constructions.
  $\Rightarrow$ For MSSM, need one **massless vector-like pair** to accommodate the **Higgs**.
- F-theory is cool. [Review in two slides.]
  $\Rightarrow$ **Quadrillion F-theory standard models (QSMs)**. [Cvetič Halverson Lin Liu Tian '19]

Goal of this talk:

Compute vector-like spectra in reps. $(\overline{\mathbf{3}}, \mathbf{2})_{1/6}$, $(\overline{\mathbf{3}}, \mathbf{1})_{-2/3}$, $(\mathbf{1}, \mathbf{1})_{1}$ of F-theory QSMs.

(Sadly, $(\overline{\mathbf{1}}, \mathbf{2})_{-1/2}$ is currently too hard for our techniques. We hope to get there in the future.)

Challenge

In global F-theory compactifications, vector-like spectra are **non-topological**.

[M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. Mayrhofer Weigand '18]

Tool

Root bundles (genearlizations of spin bundles) on **nodal curves**.

Review: SM constructions in String theory

Gauge group and chiral spectrum of SM from ST

- $E_8 \times E_8$ [Candelas Horowitz Strominger Witten '85], [Greene Kirklin Miron Ross '86], [Braun He Ovrut Pantev '05], [Bouchard Donagi '05], [Bouchard Cvetič Donagi '06], [Anderson Gray He Lukas '10], . . . , [Abel Constantin Harvey Lukas '22], . . .
- Type II [Berkooz Douglas Leigh '96], [Aldazabal Franco Ibanez Rabadan Uranga '00], [Ibanez Marchesano Rabadan '00], [Blumenhagen Kors Lust Ott '01], [Cvetič Shiu Uranga '01], . . .
- F-theory [Krause Mayrhofer Weigand '12], [Cvetič Klevers Pena Oehlmann Reuter '15], [Lin Weigand '16], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], . . .

Review: SM constructions in String theory

Gauge group and chiral spectrum of SM from ST

- $E_8 \times E_8$ [Candelas Horowitz Strominger Witten '85], [Greene Kirklin Miron Ross '86], [Braun He Ovrut Pantev '05], [Bouchard Donagi '05], [Bouchard Cvetič Donagi '06], [Anderson Gray He Lukas '10], . . . , [Abel Constantin Harvey Lukas '22], . . .
- Type II [Berkooz Douglas Leigh '96], [Aldazabal Franco Ibanez Rabadan Uranga '00], [Ibanez Marchesano Rabadan '00], [Blumenhagen Kors Lust Ott '01], [Cvetič Shiu Uranga '01], . . .
- F-theory [Krause Mayrhofer Weigand '12], [Cvetič Klevers Pena Oehlmann Reuter '15], [Lin Weigand '16], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], . . .

Gauge group, chiral and vector-like spectrum of SM from ST

- Why vector-like spectra? Higgs fields matter & characteristic feature of QFTs
- Heterotic $E_8 \times E_8$:

  [Bouchard Donagi '05], [Bouchard Cvetič Donagi '06], [Anderson Gray Lukas Palti '10 & '11], . . . , [Abel Constantin Harvey Lukas '22], . . .
- F-theory: [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18], [M.B. Cvetič Donagi Lin Liu Ruehle '20], [M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22]

F-theory is cool! [Vafa '96], ... See [Weigand '18] for a recent review.

1. Description of strongly coupled (in g_S) IIB-string theory.

F-theory is cool! [Vafa '96], ... See [Weigand '18] for a recent review.

1. Description of strongly coupled (in g_S) IIB-string theory.
2. Geometrizes physics beautifully in singular elliptic 4-fold $\pi\colon Y_4 \twoheadrightarrow B_3$.

$\Rightarrow$ Rich dictionary between physics and geometry:

- Singularity types of elliptic fibre $\leftrightarrow$ gauge groups,
- Singularity loci in B_3 $\leftrightarrow$ (intersections of) 7-brane loci,
- Consistent geometry $\leftrightarrow$ global consistency checks for physics,
- ...

F-theory is cool! [Vafa '96], ... See [Weigand '18] for a recent review.

1. Description of strongly coupled (in g_S) IIB-string theory.
2. Geometrizes physics beautifully in singular elliptic 4-fold $\pi\colon Y_4 \twoheadrightarrow B_3$.

$\Rightarrow$ Rich dictionary between physics and geometry:

- Singularity types of elliptic fibre $\leftrightarrow$ gauge groups,
- Singularity loci in B_3 $\leftrightarrow$ (intersections of) 7-brane loci,
- Consistent geometry $\leftrightarrow$ global consistency checks for physics,
- ...

3. Backbone of many field theory constructions. ... [Heckman Morrison Vafa '14], [Del Zotto Heckman Tomasiello Vafa '15], [Heckman Morrison Rudelius Vafa '15], [Schäfer-Nameki Weigand '16], [Couzens Lawrie Martelli Schäfer-Nameki Wong '17], [Bhardwaj Morrison Tachikawa Tomasiello '18], [Apruzzi Lin Mayrhofer '18], [Apruzzi Lawrie Lin Schäfer-Nameki Wang '19], ...

F-theory is cool! [Vafa '96], . . . See [Weigand '18] for a recent review.

1. Description of strongly coupled (in g_S) IIB-string theory.
2. Geometrizes physics beautifully in singular elliptic 4-fold $\pi\colon\, Y_4 \twoheadrightarrow B_3$.

$\Rightarrow$ Rich dictionary between physics and geometry:

- Singularity types of elliptic fibre $\leftrightarrow$ gauge groups,
- Singularity loci in B_3 $\leftrightarrow$ (intersections of) 7-brane loci,
- Consistent geometry $\leftrightarrow$ global consistency checks for physics,
- . . .

3. Backbone of many field theory constructions. . . . [Heckman Morrison Vafa '14], [Del Zotto Heckman Tomasiello Vafa '15], [Heckman Morrison Rudelius Vafa '15], [Schäfer-Nameki Weigand '16], [Couzens Lawrie Martelli Schäfer-Nameki Wong '17], [Bhardwaj Morrison Tachikawa Tomasiello '18], [Apruzzi Lin Mayrhofer '18], [Apruzzi Lawrie Lin Schäfer-Nameki Wang '19], . . .
4. F-theory QSMs [Cvetič Halverson Lin Liu Tian '19]:
   Largest currently-known class of string theory SM-constructions with:

   Global consistency, gauge coupling unification, no chiral exotics.

F-theory is cool! [Vafa '96], ... See [Weigand '18] for a recent review.

1. Description of strongly coupled (in g_S) IIB-string theory.
2. Geometrizes physics beautifully in singular elliptic 4-fold $\pi\colon Y_4 \twoheadrightarrow B_3$.

$\Rightarrow$ Rich dictionary between physics and geometry:

- Singularity types of elliptic fibre $\leftrightarrow$ gauge groups,
- Singularity loci in B_3 $\leftrightarrow$ (intersections of) 7-brane loci,
- Consistent geometry $\leftrightarrow$ global consistency checks for physics,
- ...

3. Backbone of many field theory constructions. ... [Heckman Morrison Vafa '14], [Del Zotto Heckman Tomasiello Vafa '15], [Heckman Morrison Rudelius Vafa '15], [Schäfer-Nameki Weigand '16], [Couzens Lawrie Martelli Schäfer-Nameki Wong '17], [Bhardwaj Morrison Tachikawa Tomasiello '18], [Apruzzi Lin Mayrhofer '18], [Apruzzi Lawrie Lin Schäfer-Nameki Wang '19], ...
4. F-theory QSMs [Cvetič Halverson Lin Liu Tian '19]:
   Largest currently-known class of string theory SM-constructions with:

   Global consistency, gauge coupling unification, no chiral exotics.

In This talk ...

Investigate vector-like spectra in **F-theory QSMs**.

Chiral and desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\#\ n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$			
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$			
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$			
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$			
$C_{(\mathbf{1},\mathbf{1})_{1}} = V(s_1, s_5)$			
How to compute?			

Chiral and desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} =$ # chiral fields in rep $\mathbf{R}$	# $n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$			3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$			3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$			3
How to compute?			

Chiral and desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \ n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$			3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$			3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$			3
How to compute?			$\chi = \int_{S_{\mathbf{R}}} G_4$

Chiral and desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \ n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$			3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$			3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$			3
$C_{(\mathbf{1},\mathbf{1})_{1}} = V(s_1, s_5)$			3
How to compute?			$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

Chiral and desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \; n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	3	0	3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4	1	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	3	0	3
How to compute?			$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

Chiral and desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \; n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	3	0	3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4	1	3
	$(4, 1) = (3, 0) \oplus (1, 1) =$ leptons + Higgs		
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_{1}} = V(s_1, s_5)$	3	0	3
How to compute?			$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

Chiral and desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}}$ = # chiral fields in rep $\mathbf{R}$	# $n_{\overline{\mathbf{R}}}$ = chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	3	0	3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4 $(4,1) = (3,0) \oplus (1,1)$ = leptons + Higgs	1	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_{1}} = V(s_1, s_5)$	3	0	3
How to compute?	$h^0(C_{\mathbf{R}}, P_{\mathbf{R}})$	$h^1(C_{\mathbf{R}}, P_{\mathbf{R}})$	$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

Chiral and desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \ n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	3	0	3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4	1	3
	$(4,1) = (3,0) \oplus (1,1) =$ leptons + Higgs		
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_{1}} = V(s_1, s_5)$	3	0	3
How to compute?	$h^0(C_{\mathbf{R}}, P_{\mathbf{R}})$	$h^1(C_{\mathbf{R}}, P_{\mathbf{R}})$	$\chi = \int_{S_{\mathbf{R}}} G_4 = 3$
	[M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17]		
	[M.B. '18] and references therein		[Cvetič Halverson Lin Liu Tian '19]

Chiral and desired vector-like spectra in the QSMs

Matter curve $C_{\mathbf{R}}$	$n_{\mathbf{R}} = \#$ chiral fields in rep $\mathbf{R}$	$\# \; n_{\overline{\mathbf{R}}} =$ chiral fields in rep $\overline{\mathbf{R}}$	Chiral index $\chi = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	3	0	3
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	4	1	3
	$(4,1) = (3,0) \oplus (1,1) =$ leptons + Higgs		
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	3	0	3
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	3	0	3
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	3	0	3
How to compute?	$h^0(C_{\mathbf{R}}, P_{\mathbf{R}})$ [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17] [M.B. '18] and references therein	$h^1(C_{\mathbf{R}}, P_{\mathbf{R}})$	$\chi = \deg(P_{\mathbf{R}}) - g(C_{\mathbf{R}}) + 1$ $\chi = \int_{S_{\mathbf{R}}} G_4 = 3$ [Cvetič Halverson Lin Liu Tian '19]

Necessary condition for P: Root bundle constraints [M.B. Cvetič Donagi Liu Ong '21]

- Finding $P_{\mathbf{R}}$ is hard [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18].

$\Rightarrow$ Try with *simple* necessary conditions:

Matter curve $C_{\mathbf{R}}$	Necessary root bundle condition for $P_{\mathbf{R}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	$P^{\otimes 36}_{(\mathbf{3},\mathbf{2})_{1/6}} = K^{\otimes 24}_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}$
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	$P^{\otimes 36}_{(\mathbf{1},\mathbf{2})_{-1/2}} = K^{\otimes 22}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30 \cdot Y_1)$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	$P^{\otimes 36}_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = K^{\otimes 24}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}}$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	$P^{\otimes 36}_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = K^{\otimes 22}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}} \otimes \mathcal{O}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}}(-30 \cdot Y_3)$
$C_{(\mathbf{1},\mathbf{1})_{1}} = V(s_1, s_5)$	$P^{\otimes 36}_{(\mathbf{1},\mathbf{1})_{1}} = K^{\otimes 24}_{C_{(\mathbf{1},\mathbf{1})_{1}}}$

Constraints for base 3-folds B_3 with $K^3_{B_3} = 18$. See [M.B. Cvetič Donagi Liu Ong '21] for exponents of B_3 with other $K^3_{B_3}$.

Necessary condition for P: Root bundle constraints [M.B. Cvetič Donagi Liu Ong '21]

- Finding $P_{\mathbf{R}}$ is hard [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18].

$\Rightarrow$ Try with *simple* necessary conditions:

Matter curve $C_{\mathbf{R}}$	Necessary root bundle condition for $P_{\mathbf{R}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	$P_{(\mathbf{3},\mathbf{2})_{1/6}}^{\otimes 36} = K_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}^{\otimes 24}$
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	$P_{(\mathbf{1},\mathbf{2})_{-1/2}}^{\otimes 36} = K_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}^{\otimes 22} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30 \cdot Y_1)$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	$P_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}^{\otimes 36} = K_{C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}}^{\otimes 24}$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	$P_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}^{\otimes 36} = K_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}}^{\otimes 22} \otimes \mathcal{O}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}}(-30 \cdot Y_3)$
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	$P_{(\mathbf{1},\mathbf{1})_1}^{\otimes 36} = K_{C_{(\mathbf{1},\mathbf{1})_1}}^{\otimes 24}$

Constraints for base 3-folds B_3 with $K_{B_3}^3 = 18$. See [M.B. Cvetič Donagi Liu Ong '21] for exponents of B_3 with other $K_{B_3}^3$.

- Root bundle constraints highly non-trivial:
  Infinitely many line bundles with $\chi = 3$ but only finitely many root bundles.

Necessary condition for P: Root bundle constraints [M.B. Cvetič Donagi Liu Ong '21]

- Finding $P_{\mathbf{R}}$ is hard [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18].

$\Rightarrow$ Try with *simple* necessary conditions:

Matter curve $C_{\mathbf{R}}$	Necessary root bundle condition for $P_{\mathbf{R}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	$P_{(\mathbf{3},\mathbf{2})_{1/6}}^{\otimes 36} = K_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}^{\otimes 24}$
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	$P_{(\mathbf{1},\mathbf{2})_{-1/2}}^{\otimes 36} = K_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}^{\otimes 22} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30 \cdot Y_1)$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	$P_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}^{\otimes 36} = K_{C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}}^{\otimes 24}$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	$P_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}^{\otimes 36} = K_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}}^{\otimes 22} \otimes \mathcal{O}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}}(-30 \cdot Y_3)$
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	$P_{(\mathbf{1},\mathbf{1})_1}^{\otimes 36} = K_{C_{(\mathbf{1},\mathbf{1})_1}}^{\otimes 24}$

Constraints for base 3-folds B_3 with $K_{B_3}^3 = 18$. See [M.B. Cvetič Donagi Liu Ong '21] for exponents of B_3 with other $K_{B_3}^3$.

- Root bundle constraints highly non-trivial:
  Infinitely many line bundles with $\chi = 3$ but only finitely many root bundles.
- Must not drop common exponents ($x^2 = 2^2 \not\Rightarrow x = 2$).

Necessary condition for P: Root bundle constraints [M.B. Cvetič Donagi Liu Ong '21]

- Finding $P_{\mathbf{R}}$ is hard [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18].

$\Rightarrow$ Try with *simple* necessary conditions:

Matter curve $C_{\mathbf{R}}$	Necessary root bundle condition for $P_{\mathbf{R}}$
$C_{(\mathbf{3},\mathbf{2})_{1/6}} = V(s_3, s_9)$	$P^{\otimes 36}_{(\mathbf{3},\mathbf{2})_{1/6}} = K^{\otimes 24}_{C_{(\mathbf{3},\mathbf{2})_{1/6}}}$
$C_{(\mathbf{1},\mathbf{2})_{-1/2}} = V\left(s_3, s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6)\right)$	$P^{\otimes 36}_{(\mathbf{1},\mathbf{2})_{-1/2}} = K^{\otimes 22}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}} \otimes \mathcal{O}_{C_{(\mathbf{1},\mathbf{2})_{-1/2}}}(-30 \cdot Y_1)$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = V(s_5, s_9)$	$P^{\otimes 36}_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}} = K^{\otimes 24}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}}$
$C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = V\left(s_9, s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5)\right)$	$P^{\otimes 36}_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}} = K^{\otimes 22}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}} \otimes \mathcal{O}_{C_{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}}(-30 \cdot Y_3)$
$C_{(\mathbf{1},\mathbf{1})_1} = V(s_1, s_5)$	$P^{\otimes 36}_{(\mathbf{1},\mathbf{1})_1} = K^{\otimes 24}_{C_{(\mathbf{1},\mathbf{1})_1}}$

Constraints for base 3-folds B_3 with $K^3_{B_3} = 18$. See [M.B. Cvetič Donagi Liu Ong '21] for exponents of B_3 with other $K^3_{B_3}$.

- Root bundle constraints highly non-trivial:
  Infinitely many line bundles with $\chi = 3$ but only finitely many root bundles.
- Must not drop common exponents ($x^2 = 2^2 \not\Rightarrow x = 2$).

$\Rightarrow$ Agenda: Vector-like spectra of the QSMs from studying root bundles.

What is known about root bundles?

- **Natural to physics:** Spin bundle S satisfies $S^2 = K_C$.

What is known about root bundles?

- **Natural to physics:** Spin bundle S satisfies $S^2 = K_C$.
- **Smooth irreducible curve** C of genus g: [Griffiths Harris "Principles of algebraic geometry" '94]
  Fix $T \in \mathrm{Pic}(C)$, $r \in \mathbb{Z}_{\geq 2}$ with $r | \deg(T)$:
  - There are exactly r^{2g} line bundles $P \in \mathrm{Pic}(C)$ with $P^r = T$.
  - Theory: Obtain all roots by twist one such P with r-torsion points of $\mathrm{Jac}(C)$.
  - **Practice: Tough.** (Related: Discrete logarithm in Picard group of elliptic curve used for elliptic-curve cryptography).

What is known about root bundles?

- **Natural to physics:** Spin bundle S satisfies $S^2 = K_C$.
- **Smooth irreducible curve** C of genus g: [Griffiths Harris "Principles of algebraic geometry" '94]
  Fix $T \in \mathrm{Pic}(C)$, $r \in \mathbb{Z}_{\geq 2}$ with $r|\deg(T)$:
  - There are exactly r^{2g} line bundles $P \in \mathrm{Pic}(C)$ with $P^r = T$.
  - Theory: Obtain all roots by twist one such P with r-torsion points of $\mathrm{Jac}(C)$.
  - **Practice: Tough.** (Related: Discrete logarithm in Picard group of elliptic curve used for elliptic-curve cryptography).
- **Nodal curve** $C^\bullet$ of genus g: [Jarves '98], [Caporaso Casagrande Cornalba '04]
  Fix $T^\bullet \in \mathrm{Pic}(C^\bullet)$, $r \in \mathbb{Z}_{\geq 2}$ with $r|\deg(T^\bullet)$:
  - There are exactly r^{2g} solutions $P^\bullet$ to $(P^\bullet)^r = T^\bullet$.
  - Theory: Explicit description from bi-weighted graphs. [Caporaso Casagrande Cornalba '04]
  - **Practice: Combinatoric challenging – often doable.**

What is known about root bundles?

- **Natural to physics:** Spin bundle S satisfies $S^2 = K_C$.
- **Smooth irreducible curve** C of genus g: [Griffiths Harris "Principles of algebraic geometry" '94]
  Fix $T \in \mathrm{Pic}(C)$, $r \in \mathbb{Z}_{\geq 2}$ with $r | \deg(T)$:
  - There are exactly r^{2g} line bundles $P \in \mathrm{Pic}(C)$ with $P^r = T$.
  - Theory: Obtain all roots by twist one such P with r-torsion points of $\mathrm{Jac}(C)$.
  - **Practice: Tough.** (Related: Discrete logarithm in Picard group of elliptic curve used for elliptic-curve cryptography).
- **Nodal curve** $C^\bullet$ of genus g: [Jarves '98], [Caporaso Casagrande Cornalba '04]
  Fix $T^\bullet \in \mathrm{Pic}(C^\bullet)$, $r \in \mathbb{Z}_{\geq 2}$ with $r | \deg(T^\bullet)$:
  - There are exactly r^{2g} solutions $P^\bullet$ to $(P^\bullet)^r = T^\bullet$.
  - Theory: Explicit description from bi-weighted graphs. [Caporaso Casagrande Cornalba '04]
  - **Practice: Combinatoric challenging – often doable.**

Refined idea

Learn about the vector-like spectra of the QSMs from root bundles on **nodal** curves.

What is known about root bundles?

- **Natural to physics:** Spin bundle S satisfies $S^2 = K_C$.
- **Smooth irreducible curve** C of genus g: [Griffiths Harris "Principles of algebraic geometry" '94]
  Fix $T \in \mathrm{Pic}(C)$, $r \in \mathbb{Z}_{\geq 2}$ with $r|\deg(T)$:
  - There are exactly r^{2g} line bundles $P \in \mathrm{Pic}(C)$ with $P^r = T$.
  - Theory: Obtain all roots by twist one such P with r-torsion points of $\mathrm{Jac}(C)$.
  - **Practice: Tough.** (Related: Discrete logarithm in Picard group of elliptic curve used for elliptic-curve cryptography).
- **Nodal curve** $C^\bullet$ of genus g: [Jarves '98], [Caporaso Casagrande Cornalba '04]
  Fix $T^\bullet \in \mathrm{Pic}(C^\bullet)$, $r \in \mathbb{Z}_{\geq 2}$ with $r|\deg(T^\bullet)$:
  - There are exactly r^{2g} solutions $P^\bullet$ to $(P^\bullet)^r = T^\bullet$.
  - Theory: Explicit description from bi-weighted graphs. [Caporaso Casagrande Cornalba '04]
  - **Practice: Combinatoric challenging – often doable.**

Refined idea

Learn about the vector-like spectra of the QSMs from root bundles on **nodal** curves.

1. How does the combinatorics work?

What is known about root bundles?

- **Natural to physics:** Spin bundle S satisfies $S^2 = K_C$.
- **Smooth irreducible curve** C of genus g: [Griffiths Harris "Principles of algebraic geometry" '94]
  Fix $T \in \mathrm{Pic}(C)$, $r \in \mathbb{Z}_{\geq 2}$ with $r | \deg(T)$:
  - There are exactly r^{2g} line bundles $P \in \mathrm{Pic}(C)$ with $P^r = T$.
  - Theory: Obtain all roots by twist one such P with r-torsion points of $\mathrm{Jac}(C)$.
  - **Practice: Tough.** (Related: Discrete logarithm in Picard group of elliptic curve used for elliptic-curve cryptography).
- **Nodal curve** $C^\bullet$ of genus g: [Jarves '98], [Caporaso Casagrande Cornalba '04]
  Fix $T^\bullet \in \mathrm{Pic}(C^\bullet)$, $r \in \mathbb{Z}_{\geq 2}$ with $r | \deg(T^\bullet)$:
  - There are exactly r^{2g} solutions $P^\bullet$ to $(P^\bullet)^r = T^\bullet$.
  - Theory: Explicit description from bi-weighted graphs. [Caporaso Casagrande Cornalba '04]
  - **Practice: Combinatoric challenging – often doable.**

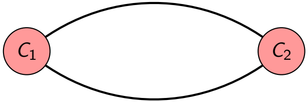
Refined idea

Learn about the vector-like spectra of the QSMs from root bundles on **nodal** curves.

1. How does the combinatorics work?
2. How do we get nodal matter curves in the QSMs?

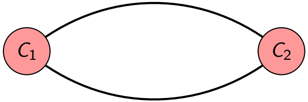
Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]

- Nodal curve: Two $\mathbb{P}^1$s – C_1, C_2 – meeting in two nodal singularities.



Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]

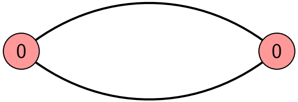
- Nodal curve: Two $\mathbb{P}^1$s – C_1, C_2 – meeting in two nodal singularities.



- Adjunction formula: $\deg(K_{C^\bullet}|_{C_i})) = -2 + (\#\text{nodes on } C_i) = 0$.

Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]

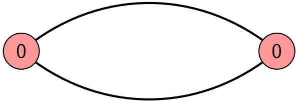
- Nodal curve: Two $\mathbb{P}^1$s – C_1, C_2 – meeting in two nodal singularities.



- Adjunction formula: $\deg(K_{C^\bullet}|_{C_i})) = -2 + (\#\text{nodes on } C_i) = 0$.

Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]

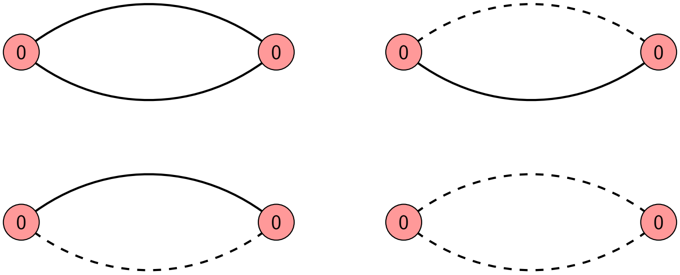
- Nodal curve: Two $\mathbb{P}^1$s – C_1, C_2 – meeting in two nodal singularities.



- Adjunction formula: $\deg(K_{C^\bullet}|_{C_i}) = -2 + (\#\text{nodes on } C_i) = 0$.
- Procedure:
  1. Pick $r \in \mathbb{Z}_{\geq 2}$ such that $r|\deg(K_{C^\bullet})$. For the following example: $r = 2$.
  2. Binary choice for each edge/nodal singularity: Blow it up or keep it.
  3. At each blown-up edge, place two weights $u, v \in \{1, 2, \ldots, r-1\}$.
  4. Check certain conditions. (Details on the next slide.)

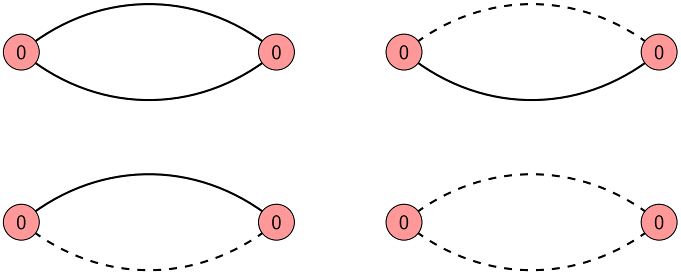
  $\Rightarrow$ Torsion-free, non locally-free sheaves $P^\bullet$ with $(P^\bullet)^{\otimes r} = K_{C^\bullet}$.

Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]



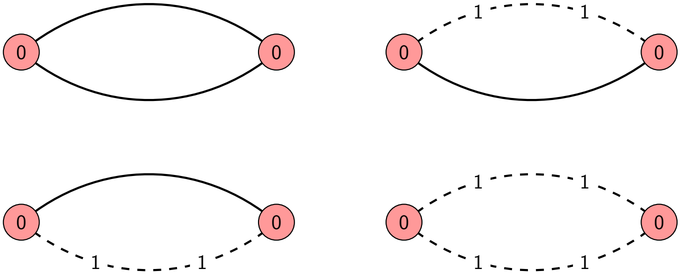
Dashed: Blown-up nodal singularity.

Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]



Dashed: Blown-up nodal singularity.

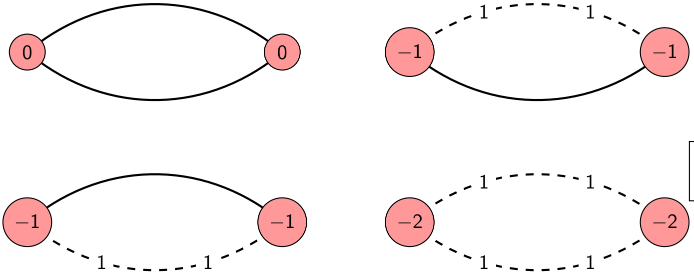
Weights $u, v \in \{1, 2, \dots, r-1\}$.
So $u, v, \in \{1\}$ for $r = 2$.



Dashed: Blown-up nodal singularity.

Weights $u, v \in \{1, 2, \ldots, r-1\}$.
So $u, v, \in \{1\}$ for $r = 2$.

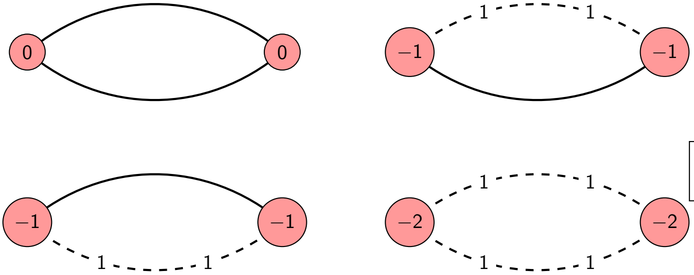
Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]



Dashed: Blown-up nodal singularity.

Weights $u, v \in \{1, 2, \ldots, r-1\}$.
So $u, v, \in \{1\}$ for $r = 2$.

Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]

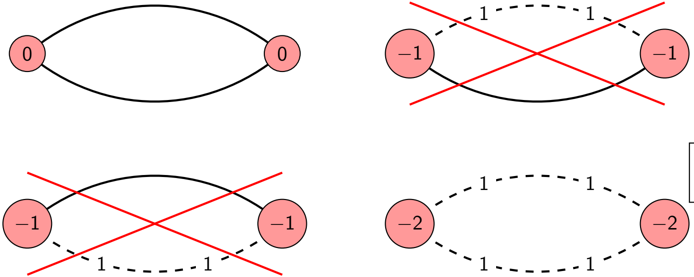


Dashed: Blown-up nodal singularity.

Weights $u, v \in \{1, 2, \ldots, r-1\}$.
So $u, v, \in \{1\}$ for $r = 2$.

- Condition: Reduced degree divisible by r

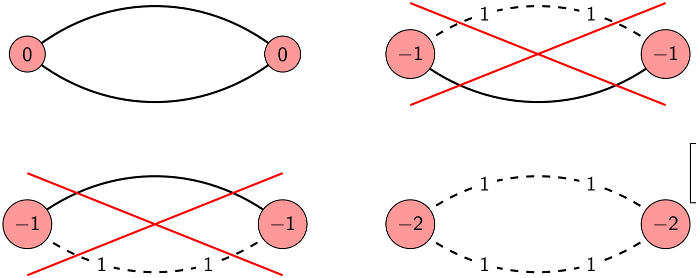
Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]



Dashed: Blown-up nodal singularity.

Weights $u, v \in \{1, 2, \ldots, r-1\}$. So $u, v, \in \{1\}$ for $r = 2$.

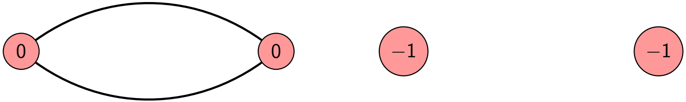
- Condition: Reduced degree divisible by r – here rules out two setups.



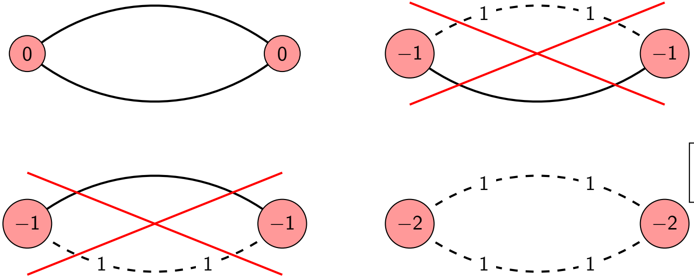
Dashed: Blown-up nodal singularity.

Weights $u, v \in \{1, 2, \ldots, r-1\}$.
So $u, v, \in \{1\}$ for $r = 2$.

- Condition: Reduced degree divisible by r – here rules out two setups.
- Divide degrees by r and find h^0 (*descent data/how sections glue across nodes*)



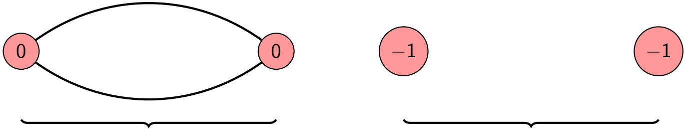
Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]



Dashed: Blown-up nodal singularity.

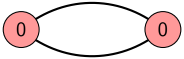
Weights $u, v \in \{1, 2, \ldots, r-1\}$.
So $u, v, \in \{1\}$ for $r = 2$.

- Condition: Reduced degree divisible by r – here rules out two setups.
- Divide degrees by r and find h^0 (*descent data/how sections glue across nodes*)

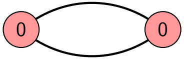


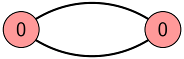
$h^0 = 1$ $h^0 = 0$

Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]

Smooth $g = 1$ curve h^0 of spin bundle P		Nodal curve $C^\bullet$ Limit root $P^\bullet$	h^0	Multiplicity
1 0	$\leftarrow$ Deformation $\rightarrow$		1	
0 0	$\leftarrow$ Deformation $\rightarrow$		0	

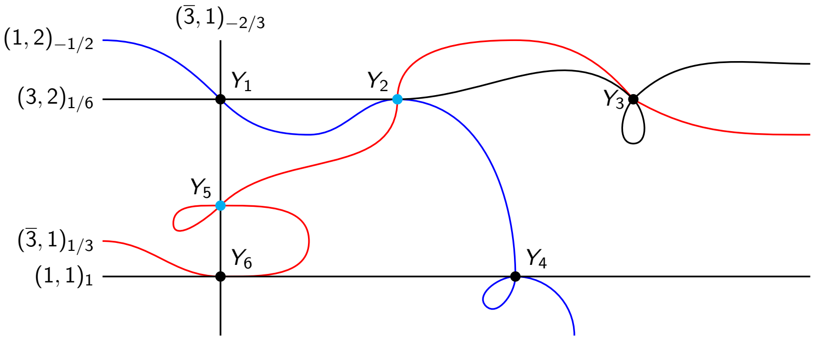
Example: Spin bundles on simple nodal curve [Caporaso Casagrande Cornalba '04]

Smooth $g = 1$ curve		Nodal curve $C^\bullet$		
h^0 of spin bundle P		Limit root $P^\bullet$	h^0	Multiplicity
1 0	$\leftarrow$ Deformation $\rightarrow$		1	$\mu = r^{b_1} = 2$
0 0	$\leftarrow$ Deformation $\rightarrow$		0	$\mu = r^{b_1} = 2$

Smooth $g = 1$ curve h^0 of spin bundle P		Nodal curve $C^\bullet$		
		Limit root $P^\bullet$	h^0	Multiplicity
1 0	$\leftarrow$ Deformation $\rightarrow$		1	$\mu = r^{b_1} = 2$
0 0	$\leftarrow$ Deformation $\rightarrow$		0	$\mu = r^{b_1} = 2$

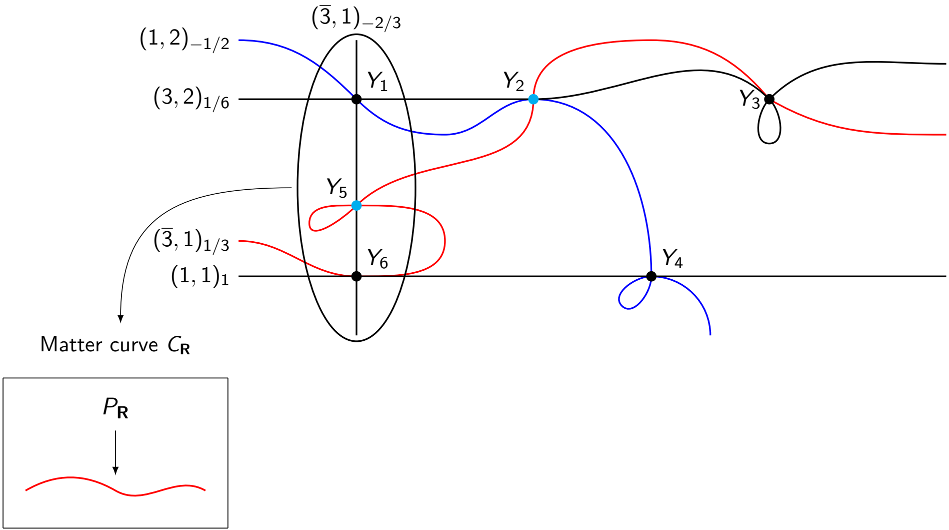
Upper semi-continuity

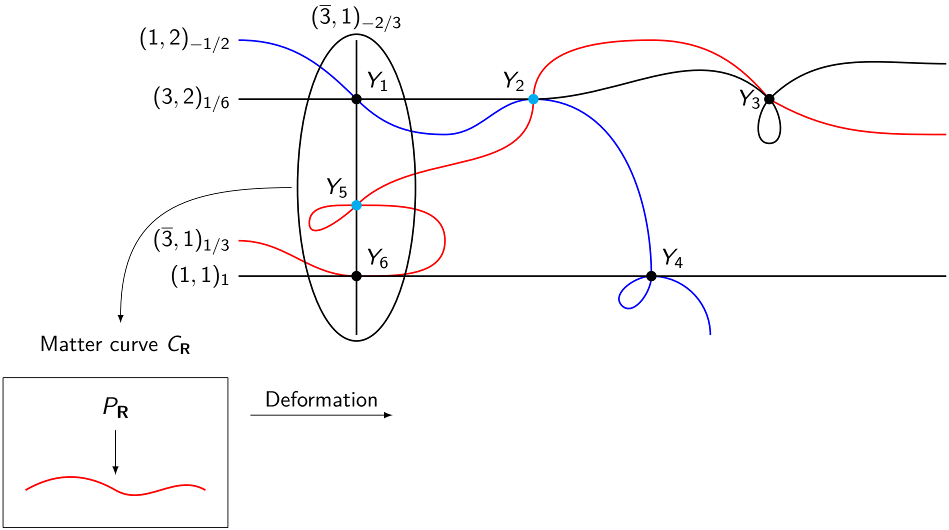
$$h^0(C^\bullet, P^\bullet) \geq h^0(C, P)$$



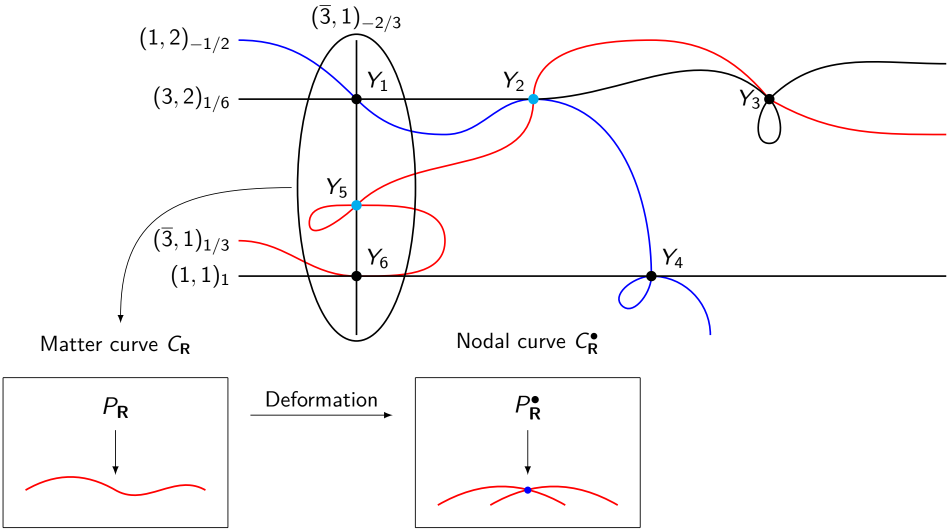
$(\overline{3}, 1)_{-2/3}$
$(1, 2)_{-1/2}$
$(3, 2)_{1/6}$
$(\overline{3}, 1)_{1/3}$
$(1, 1)_1$
Y_1
Y_2
Y_3
Y_4
Y_5
Y_6

Philosophy: Local, bottom-up . . . [M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22]

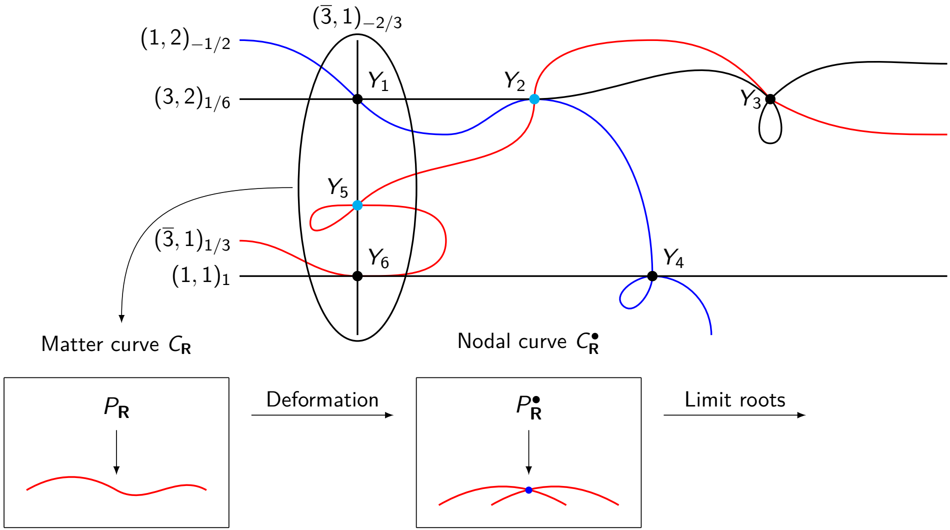




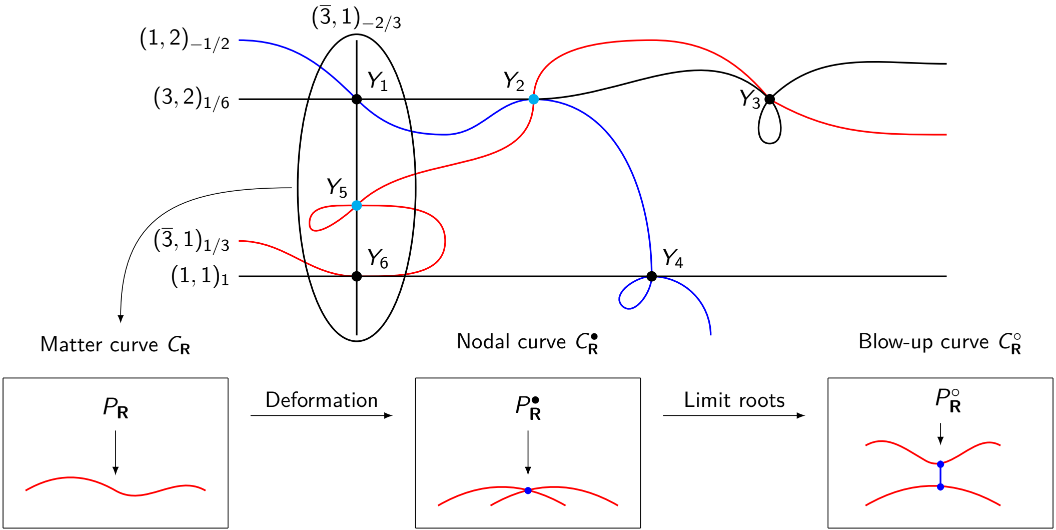
$(\overline{3},1)_{-2/3}$
$(1,2)_{-1/2}$
$(3,2)_{1/6}$
$(\overline{3},1)_{1/3}$
$(1,1)_1$
Y_1
Y_2
Y_3
Y_4
Y_5
Y_6
Matter curve $C_\mathbf{R}$
$P_\mathbf{R}$
Deformation



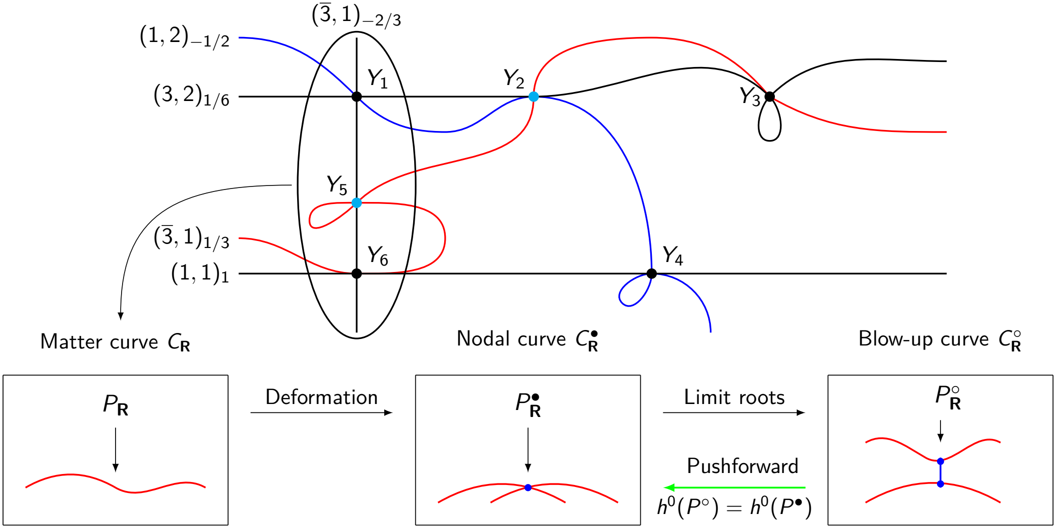
$(\overline{3}, 1)_{-2/3}$
$(1, 2)_{-1/2}$
$(3, 2)_{1/6}$
$(\overline{3}, 1)_{1/3}$
$(1, 1)_1$
Y_1
Y_2
Y_3
Y_4
Y_5
Y_6
Matter curve $C_{\mathbf{R}}$
Nodal curve $C^\bullet_{\mathbf{R}}$
$P_{\mathbf{R}}$
Deformation
$P^\bullet_{\mathbf{R}}$

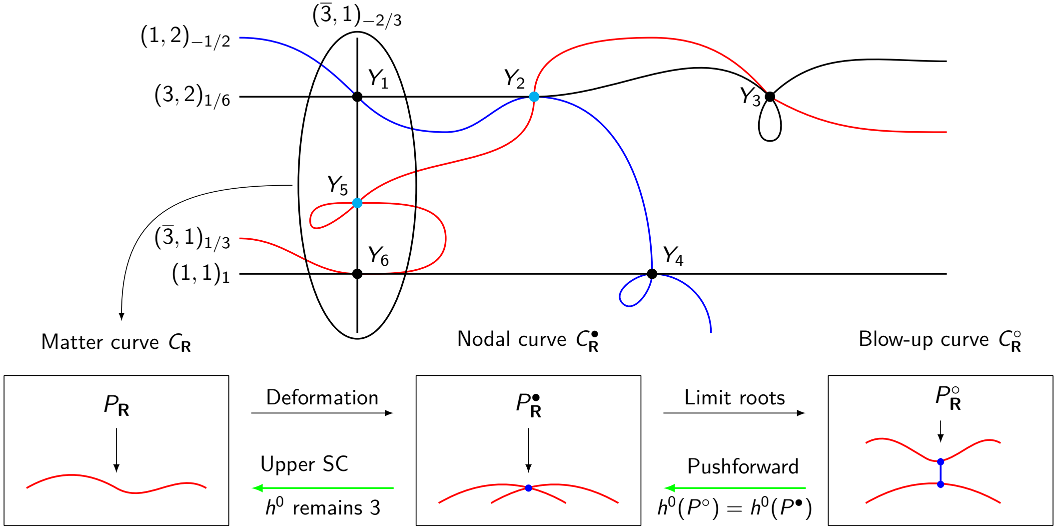


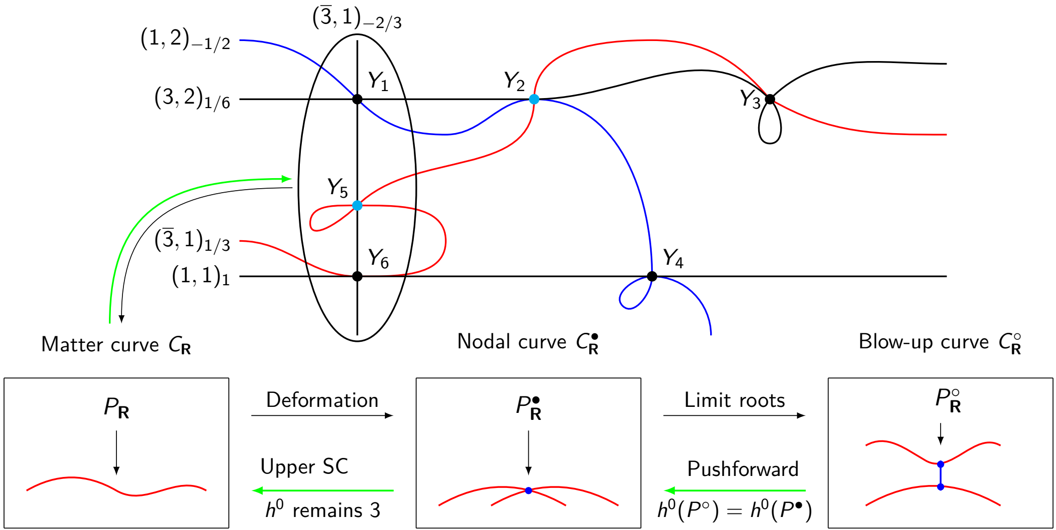
$(\overline{3},1)_{-2/3}$
$(1,2)_{-1/2}$
$(3,2)_{1/6}$
$(\overline{3},1)_{1/3}$
$(1,1)_1$
Y_1
Y_2
Y_3
Y_4
Y_5
Y_6
Matter curve $C_{\mathbf{R}}$
Nodal curve $C_{\mathbf{R}}^\bullet$
$P_{\mathbf{R}}$
Deformation
$P_{\mathbf{R}}^\bullet$
Limit roots


$(\overline{3}, 1)_{-2/3}$
$(1, 2)_{-1/2}$
$(3, 2)_{1/6}$
$(\overline{3}, 1)_{1/3}$
$(1, 1)_1$
Y_1
Y_2
Y_3
Y_4
Y_5
Y_6
Matter curve $C_{\mathbf{R}}$
Nodal curve $C^{\bullet}_{\mathbf{R}}$
Blow-up curve $C^{\circ}_{\mathbf{R}}$
$P_{\mathbf{R}}$
Deformation
$P^{\bullet}_{\mathbf{R}}$
Limit roots
$P^{\circ}_{\mathbf{R}}$

Philosophy: Local, bottom-up . . . [M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22]




$(\overline{3}, 1)_{-2/3}$
$(1, 2)_{-1/2}$
$(3, 2)_{1/6}$
$(\overline{3}, 1)_{1/3}$
$(1, 1)_1$
Y_1
Y_2
Y_3
Y_4
Y_5
Y_6
Matter curve $C_{\mathbf{R}}$
Nodal curve $C^\bullet_{\mathbf{R}}$
Blow-up curve $C^\circ_{\mathbf{R}}$
$P_{\mathbf{R}}$
Deformation
Upper SC
h^0 remains 3
$P^\bullet_{\mathbf{R}}$
Limit roots
Pushforward
$h^0(P^\circ) = h^0(P^\bullet)$
$P^\circ_{\mathbf{R}}$

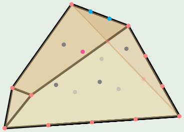


$(\overline{3}, 1)_{-2/3}$
$(1, 2)_{-1/2}$
$(3, 2)_{1/6}$
$(\overline{3}, 1)_{1/3}$
$(1, 1)_1$
Y_1
Y_2
Y_3
Y_4
Y_5
Y_6
Matter curve $C_{\mathbf{R}}$
Nodal curve $C^{\bullet}_{\mathbf{R}}$
Blow-up curve $C^{\circ}_{\mathbf{R}}$
$P_{\mathbf{R}}$
Deformation
Upper SC
h^0 remains 3
$P^{\bullet}_{\mathbf{R}}$
Limit roots
Pushforward
$h^0(P^{\circ}) = h^0(P^{\bullet})$
$P^{\circ}_{\mathbf{R}}$

Philosophy: Local, bottom-up and FRST invariant

[M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22]

Advantage: Triangulation invariant estimate of VL spectra for huge families of QSMs



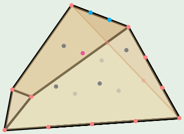
Δ° ⟶ fine regular star triangulations ⟶ Family $B_3(\Delta^\circ)$ of toric F-theory base 3-folds - - - ► **Same** nodal matter curve $C^\bullet_{\mathbf{R}}$ $\forall X_\Sigma \in B_3(\Delta^\circ)$

[Kreuzer Skarke '98], [Klevers Peña Oehlmann Piragua Reuter '14], [Cvetič Klevers Peña Oehlmann Reuter '15], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], . . .

Philosophy: Local, bottom-up and FRST invariant

[M.B. Cvetič Donagi Liu Ong '21], [M.B. Cvetič Liu '21], [M.B. Cvetič Donagi Ong '22]

Advantage: Triangulation invariant estimate of VL spectra for huge families of QSMs



Δ° ⟶ fine regular star triangulations ⟶ Family $B_3(\Delta^\circ)$ of toric F-theory base 3-folds - - -► **Same** nodal matter curve $C_{\mathbf{R}}^{\bullet}$ $\forall X_\Sigma \in B_3(\Delta^\circ)$

[Kreuzer Skarke '98], [Klevers Peña Oehlmann Piragua Reuter '14], [Cvetič Klevers Peña Oehlmann Reuter '15], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], . . .

Interlude: Computer algebra systems

- Triangulations in [M.B. Cvetič Donagi Ong '22] done with the modern computer algebra system `OSCAR`, which – due to the use of the `Julia` programming language – is expected to be very performant.
- For *fast* triangulations, also look at `CY-Tools` [Liam McAllister group], which hopefully can be available via `OSCAR` soon.

Towards "good" physical roots

(Naive) Brill-Noether theory for root bundles

Discriminate the r^{2g} limit roots $P^\bullet$ with $(P^\bullet)^{\otimes r} = T$ according to $h^0(C^\bullet, P^\bullet)$:

$$r^{2g} = N_0 + N_1 + N_2 + \ldots \, , \tag{1}$$

where N_i is the number of limit roots with $h^0(C^\bullet, P^\bullet) = i$.

Towards "good" physical roots

(Naive) Brill-Noether theory for **root bundles**

Discriminate the r^{2g} limit roots $P^\bullet$ with $(P^\bullet)^{\otimes r} = T$ according to $h^0(C^\bullet, P^\bullet)$:

$$r^{2g} = N_0 + N_1 + N_2 + \ldots \,, \tag{1}$$

where N_i is the number of limit roots with $h^0(C^\bullet, P^\bullet) = i$.

Current standing

- Systematic answer unknown (to my knowledge).
- For sufficiently simple setups can count N_i, **but**:
  - Ignorance: Currently, we can sometimes only compute a lower bound to h^0.
  - Jumping circuits: h^0 can jump if nodes are specially aligned. [M.B. Cvetič Donagi Ong '22]

  $\Rightarrow$ Denote the number of these cases by $\widetilde{N}_{\geq i}$.

$$r^{2g} = \left(\widetilde{N}_0 + \widetilde{N}_{\geq 0}\right) + \left(\widetilde{N}_1 + \widetilde{N}_{\geq 1}\right) + \ldots \,. \tag{2}$$

Brill-Noether numbers of $(\overline{\mathbf{3}}, \mathbf{2})_{1/6}$ in QSMs with $\overline{K}_{B_3}^3 = 6$

- First estimates computed in [M.B. Cvetič Liu '21]:
  - count **"simple"** root bundles with minimal h^0,
  - no estimate for $\widetilde{N}_{\geq i}$.
- Refinements/extensions in [M.B. Cvetič Donagi Ong '22]:
  - count **all** root bundles,
  - discriminate via line bundle cohomology on rational tree-like nodal curves.

QSM-family (KS polytope)	# FRSTs	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$
Δ_8°	$\sim 10^{15}$	57.3%	?	?	?
Δ_4°	$\sim 10^{11}$	53.6%	?	?	?
Δ_{134}°	$\sim 10^{10}$	48.7%	?	?	?
Δ_{128}°, Δ_{130}°, Δ_{136}°, Δ_{236}°	$\sim 10^{11}$	42.0%	?	?	?

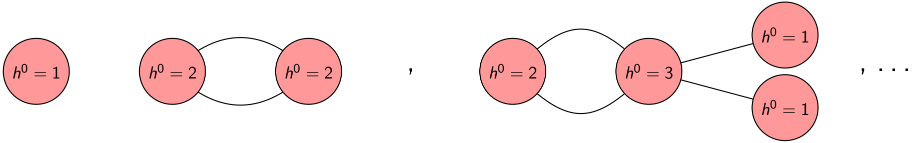
Brill-Noether numbers of $(\overline{\mathbf{3}}, \mathbf{2})_{1/6}$ in QSMs with $\overline{K}_{B_3}^3 = 6$

- First estimates computed in [M.B. Cvetič Liu '21]:
  - count **"simple"** root bundles with minimal h^0,
  - no estimate for $\widetilde{N}_{\geq i}$.
- Refinements/extensions in [M.B. Cvetič Donagi Ong '22]:
  - count **all** root bundles,
  - discriminate via line bundle cohomology on rational tree-like nodal curves.

QSM-family (KS polytope)	# FRSTs	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$
Δ_8°	$\sim 10^{15}$	76.4%	23.6%		
Δ_4°	$\sim 10^{11}$	99.0%	1.0%		
Δ_{134}°	$\sim 10^{10}$	99.8%	0.2%		
Δ_{128}°, Δ_{130}°, Δ_{136}°, Δ_{236}°	$\sim 10^{11}$	99.9%	0.1%		

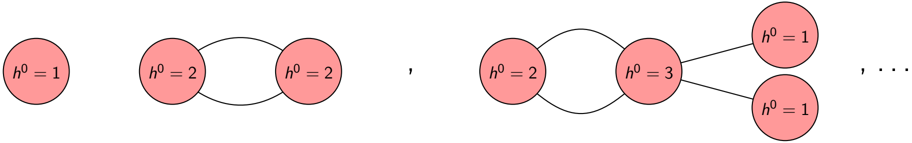
Can we do better for $B_3(\Delta_4^\circ)$? The 1% contains ...

- Stationary circuits with $h^0 = 3$:

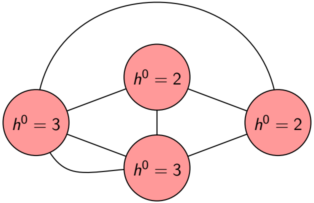


Can we do better for $B_3(\Delta_4^\circ)$? The 1% contains ...

- Stationary circuits with $h^0 = 3$:

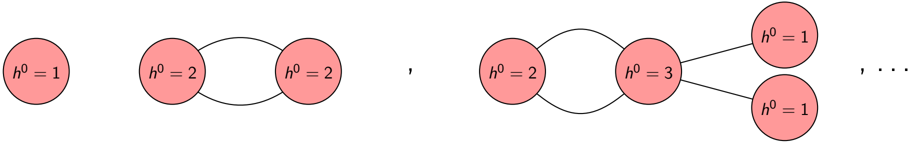


- Jumping circuit with $h^0 = 4$:

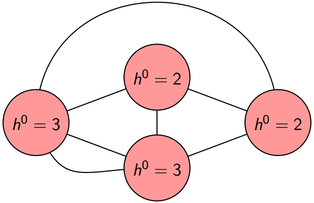


Can we do better for $B_3(\Delta_4^\circ)$? The 1% contains ...

- Stationary circuits with $h^0 = 3$:



- Jumping circuit with $h^0 = 4$:



Mistake in first preprint [M.B. Cvetič Donagi Ong '22]

- We **wrongly** computed h^0 for the jumping circuit. Correction on the `ArXiV`.

$\Rightarrow$ $B_3(\Delta_4^\circ)$: 99.995% of solutions to **necessary** root bundle constraint have $h^0 = 3$.

Brill-Noether numbers of $(\overline{3}, 2)_{1/6}$ in QSMs with $\overline{K}_{B_3}^3 = 10$ [M.B. Cvetič Donagi Ong '22]

QSM-family (polytope)	$h^0 = 3$	$h^0 \geq 3$	$h^0 = 4$	$h^0 \geq 4$	$h^0 = 5$	$h^0 \geq 5$	$h^0 = 6$	$h^0 \geq 6$
Δ°_{88}	74.9	22.1	2.5	0.5	0.0	0.0		
Δ°_{110}	82.4	14.1	3.1	0.4	0.0			
Δ°_{272}, Δ°_{274}	78.1	18.0	3.4	0.5	0.0	0.0		
Δ°_{387}	73.8	21.9	3.5	0.7	0.0	0.0		
Δ°_{798}, Δ°_{808}, Δ°_{810}, Δ°_{812}	77.0	17.9	4.4	0.7	0.0	0.0		
Δ°_{254}	95.9	0.5	3.5	0.0	0.0	0.0		
Δ°_{52}	95.3	0.7	3.9	0.0	0.0	0.0		
Δ°_{302}	95.9	0.5	3.5	0.0	0.0			
Δ°_{786}	94.8	0.3	4.8	0.0	0.0	0.0		
Δ°_{762}	94.8	0.3	4.9	0.0	0.0	0.0		
Δ°_{417}	94.8	0.3	4.8	0.0	0.0	0.0	0.0	
Δ°_{838}	94.7	0.3	5.0	0.0	0.0	0.0		
Δ°_{782}	94.6	0.3	5.0	0.0	0.0	0.0		
Δ°_{377}, Δ°_{499}, Δ°_{503}	93.4	0.2	6.2	0.0	0.1	0.0		
Δ°_{1348}	93.7	0.0	6.2	0.0	0.1		0.0	
Δ°_{882}, Δ°_{856}	93.4	0.3	6.2	0.0	0.1	0.0	0.0	
Δ°_{1340}	92.3	0.0	7.6	0.0	0.1		0.0	
Δ°_{1879}	92.3	0.0	7.5	0.0	0.1		0.0	
Δ°_{1384}	90.9	0.0	8.9	0.0	0.2		0.0	

Summary and outlook

- **Statistical observation**:
  In QSMs, absence of vector-like exotics in $(\overline{\mathbf{3}}, \mathbf{2})_{1/6}$, $(\overline{\mathbf{3}}, \mathbf{1})_{-2/3}$, $(\mathbf{1}, \mathbf{1})_1$ likely,
  but ...
  - **Sufficient** condition for quantization of G_4-flux? [Jefferson Taylor Turner '21].
  - F-theory gauge potential
    - may select (proper) subset of these root bundles,
    - lead to correlated choices on distinct matter curves.
- Vector-like spectra on $C_{\mathbf{R}}^{\bullet}$ "upper bound" to those on $C_{\mathbf{R}}$.
  $\leftrightarrow$ Understand "drops" from **Yukawa interactions**? [Cvetič Lin Liu Zhang Zoccarato '19]
  $\rightarrow$ Towards the Higgs ...
- Brill-Noether numbers on Higgs curve currently computationally too challenging.
  - Need **Brill-Noether theory for root bundles on nodal curves**.
    Map from (dual) graphs (and a couple of integers) to Brill-Noether numbers.
    $\leftrightarrow$ Arena for **machine learning**?

$\rightarrow$ **Probability/statistics** for F-theory setups to arise **without vector-like exotics**.

Thank you for your attention!