Operator growth and Krylov construction in an open system

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- Based on work done with Pratik Nandy, Pingal Pratyush Nath and Himanshu Sahu (CHEP, IISc Bangalore). arXiv: 2207.xxxxx (appeared today in arXiv)

Outline

- Introduction
- Open systems and Lindblad evolution
- Krylov iteration with Lindbladian
- Results
- Conclusion

Introduction

- Studying the evolution of quantum systems → condensed matter, QI, HEP.
- Questions→ integrability, chaos, thermalization, OTOC → Operator growth.
- Chaotic theories → operators become complexified quickly→ faster scrambling of information.
- OTOC ~ Lyapunov exponent → saturation for chaotic cases.
- Central question → distinguish integrable and chaotic cases.

- Start from a simple local operator $O(0)$ (e.g.; one site operator).
- Hamiltonian of the quantum system → Hermitian, time independent and a k sites.
- Operator evolution →Heisenberg picture →

$$O(t) = e^{iHt}O(0)e^{-iHt} = O(0) + it[H, O(0)] + \frac{i^2t^2}{2!}[H, [H, O(0)]] + \cdots$$

- Also treated as Liouvillian evolution $O(t) = e^{it\mathcal{L}}O(0),\ \mathcal{L} = [H, \cdot\ ]$
- The iterative action of Liouvillian does not generate orthonormal vectors (super operator→operator, operator → state, norm→ thermal Wightman products)

- Algorithm to construct an orthonormal basis→Krylov-Lanczos algorithm.
- The basis formed→ Krylov basis →dimension $\mathscr{K} \leq D^2 - D + 1$, with D the Hilbert space dimension.
- $\mathscr{L}|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$, where $|\mathcal{O}_n)$ are the basis elements and b_n are called the Lanczos coefficients.
- Liouvillian has a tridiagonal matrix form

$$\mathscr{L}_{mn} = (O_m|\mathscr{L}|O_n) \quad \Rightarrow \quad \mathscr{L} = \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 \\ b_1 & 0 & b_2 & \cdots & 0 \\ 0 & b_2 & 0 & b_3 & \cdots \\ \cdots & \cdots & b_3 & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & b_n \\ 0 & 0 & \cdots & b_n & 0 \end{pmatrix}.$$

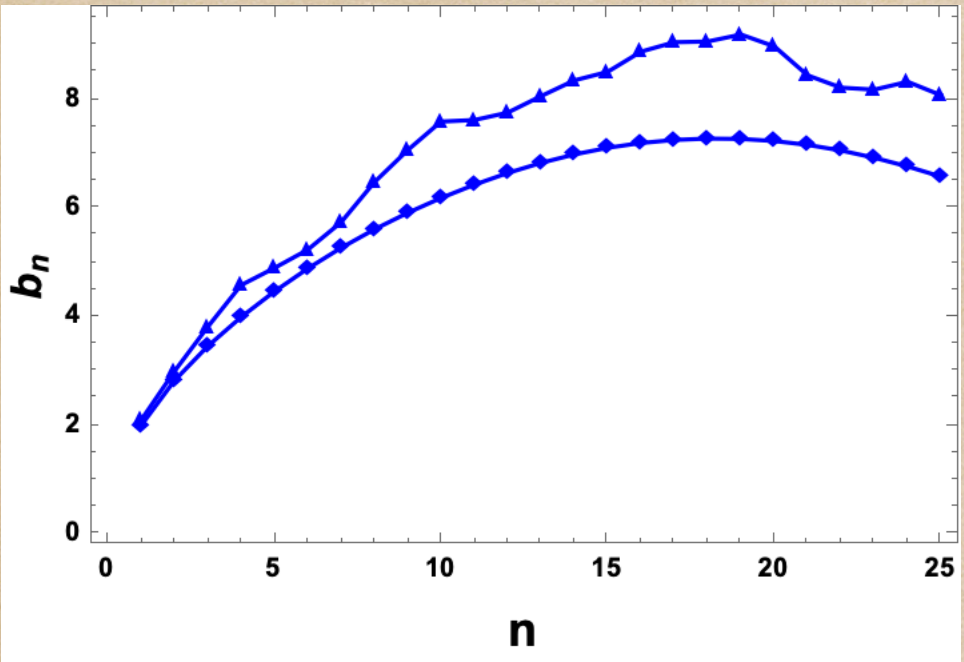
- UOGH (Universal operator growth hypothesis) → b_n growth can characterise chaos and integrability. (1812.08657, PRX.9.041.017, Parker et al)

- General behaviour of b_n → initial growth characterising growing support of the operator, then comes down to zero after exploring the whole Krylov basis.

- How an operator spreads in the Krylov basis → chaotic evolution → the initial growth is quickest (linear).

- Coming down to zero, more fluctuations implies integrability. (2112.12128, JHEP03(2022)211, Rabinovici et al)

- Level statistics of unfolded spectrum for closed systems: integrable → Poisson, chaotic → Wigner-Dyson statistics → match with results from Krylov Lanczos algorithm.



b_n
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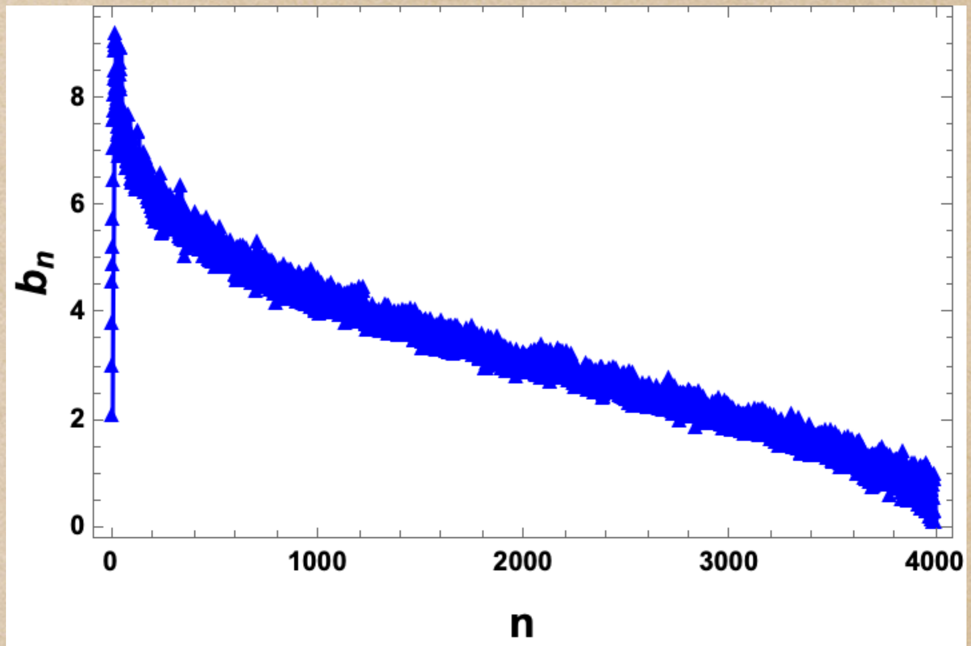
20
25



b_n
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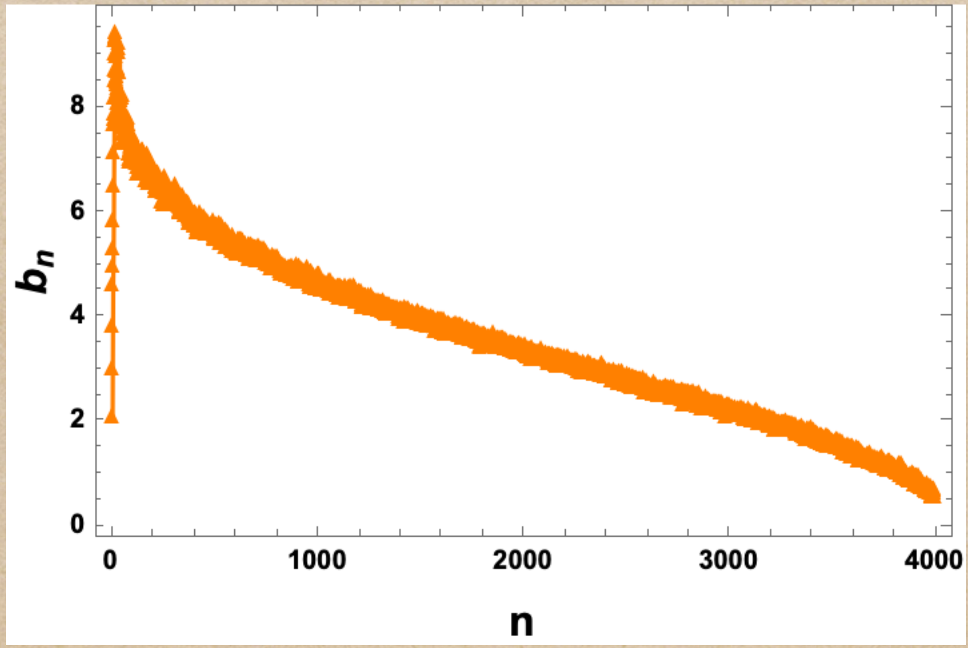
3000
4000



b_n
n
0
2
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1000
2000
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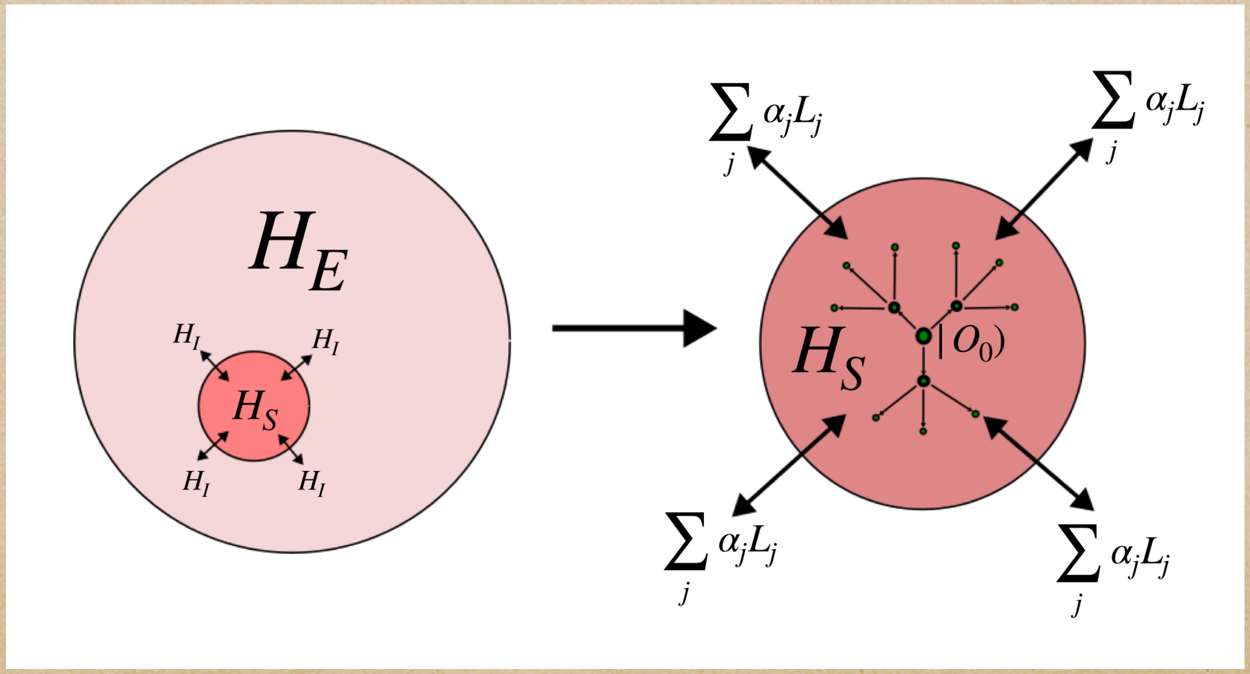
4000

- After Krylov basis is formed→ $|O_t) = \sum_{n=0}^{\mathcal{K}-1} i^n \phi_n(t) |\mathcal{O}_n)$, $|\phi_n(t)|^2$ are probabilities of finding operator in n-th Krylov basis at time t→ $\sum_{n=0}^{\mathcal{K}-1} |\phi_n|^2 = 1$.
- Average position of the operator in Krylov space, K-complexity: $C(t) = \sum_{n=0}^{\mathcal{K}-1} n|\phi_n|^2$
- K-complexity: grows exponentially for linear growth of b_n, grows linearly for saturation of b_n and saturates for the long time decrease of b_n.
- For integrable case, the saturation is at a lower value due to more fluctuations in b_n. (2112.12128, Rabinovici et al)

Open systems and Lindblad evolution

- Why open systems: pragmatic approach, preparation of ideal closed systems is impossible, always some interaction with environment.
- Evolution becomes non-unitary. Liouvillian has non-hermitian contributions→ Lindbladian.
- What happens to operator evolution and integrability in case of non-unitary evolution?
- Primary spread of operator always within system degrees of freedom.

- $H_{\mathrm{SE}} = H_{\mathrm{S}} \otimes I_{\mathrm{E}} + I_{\mathrm{S}} \otimes H_{\mathrm{E}} + H_I\,,\ H_I = \sum_i \alpha_i S_i \otimes E_i$
- S_i and E_i are operators in the system, and environment Hilbert space, respectively, α_i coupling.
- We concentrate on the evolution of a density matrix (mixed state in general) $\rho_S = Tr_E[\rho_{SE}]$.
- Assume access to the information of H_S and $\sum_i \sqrt{\alpha_i} S_i$ (system info and some understanding about the ways it interacts with environment).



- Born-Markovian approximation → Lindblad master equation

$$\dot{\rho} = -i[H,\rho] + \sum_k \left[L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right] = -i\mathscr{L}_o \rho(t)\,.$$

- Approximations: weak bath-system coupling, bath relaxation time<<system relaxation time, factorisability of total density matrix.

- $\dot{O}(t) = i\mathscr{L}_o O(t)$, with $\mathscr{L}_o[\,\bullet\,] = [H,\bullet\,] - i\sum_k \left[L_k^\dagger \bullet L_k - \frac{1}{2}\{L_k^\dagger L_k, \bullet\,\}\right]$,

- Lindbladian is non-hermitian.

- Perform similar steps? Or different? Check both.

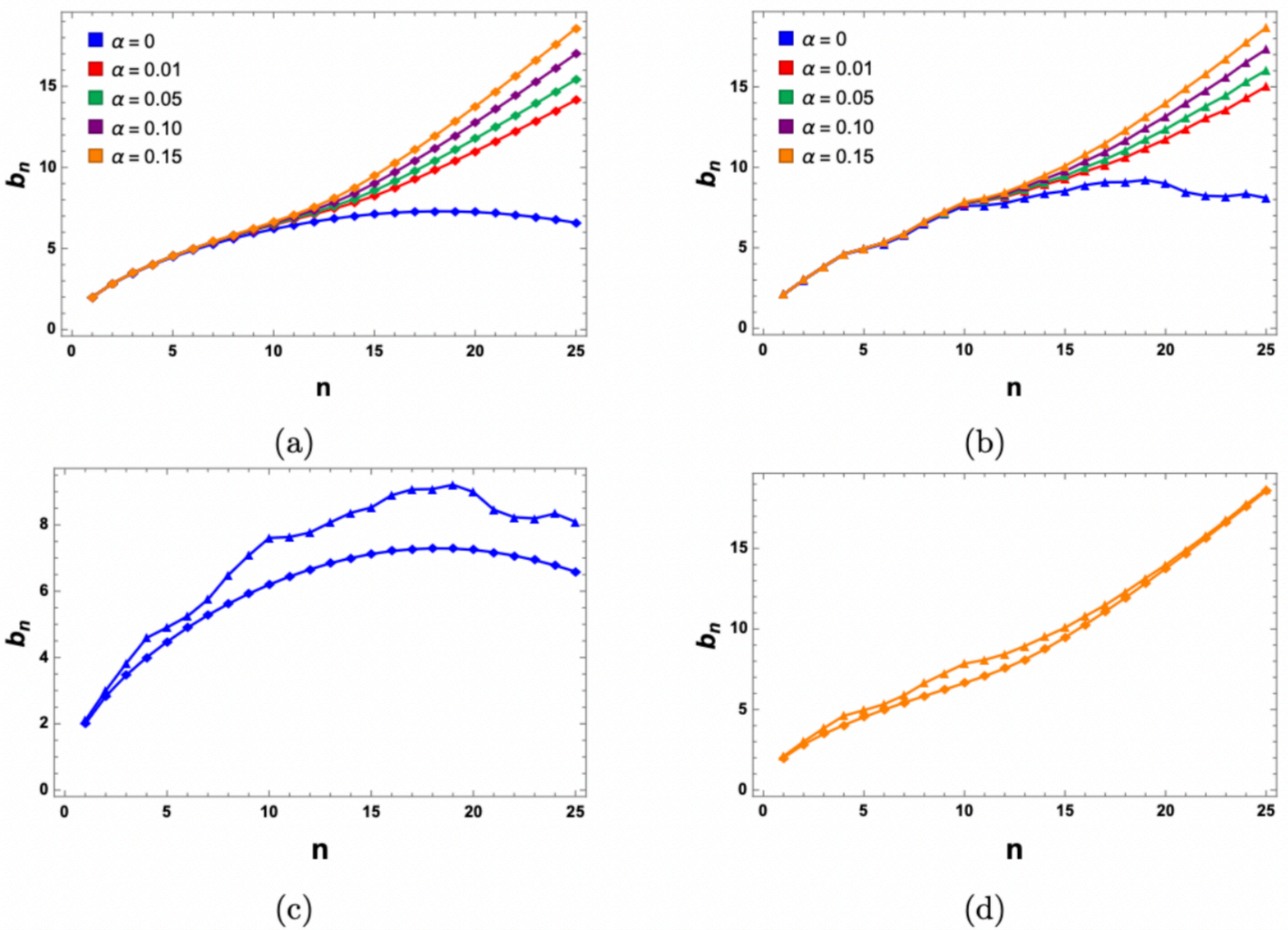
- Model: Transverse field Ising model with open BC

$$H_{\mathrm{TFIM}} = -\sum_{j=1}^{N-1} \sigma_j^z \sigma_{j+1}^z - g\sum_{j=1}^{N} \sigma_j^x - h\sum_{j=1}^{N} \sigma_j^z\,.$$

- g nonzero, $h = 0$, integrable, for nonzero h, goes away from integrability, $g = -1.05, h = 0.5$, maximally away from integrability.
- Level statistics results are there for dissipative open systems: integrable→ still Poissonian, chaotic →complex Ginibre ensemble. (1910.03520, PRL.123.254101, Akemann et al)
- Boundary Lindblad operators $L_{-1} = \sqrt{\alpha}\,\sigma_1^+, L_0 = \sqrt{\alpha}\,\sigma_1^-, L_{N+1} = \sqrt{\alpha}\,\sigma_N^+, L_{N+2} = \sqrt{\alpha}\,\sigma_N^-$.
- Bulk dephasing operators $L_i = \sqrt{\gamma}\,\sigma_i^z, \quad i = 1,2,\cdots,N$.
- Eigenvalues are real or come in complex conjugate pairs.
- Level statistics done with complex spacing ratios.

Krylov iteration with Lindbladian

- Lanczos: Applying Hermitian methods to a non-hermitian operator→observing the breakdown.
- Orthonormalisation is limited to lower accuracy→10^{-3}.
- Growth of b_n becomes unphysical once non-hermitian effects are present (keeps growing forever).
- Integrable and chaotic regimes are not distinguishable anymore.



α = 0
α = 0.01
α = 0.05
α = 0.10
α = 0.15
b_n
n
(a)
α = 0
α = 0.01
α = 0.05
α = 0.10
α = 0.15
b_n
n
(b)
b_n
n
(c)
b_n
n
(d)

- Effect of Lindblad operators (non-hermitian part) can be increased two ways, i) increasing the coupling, ii) take bulk dephasing at all sites instead of just boundary Lindblad operators.
- In both of the cases, the coefficients keep growing forever with larger and larger slope (growth rate).
- Can distinguish between different non-hermitian parameters, but can not distinguish integrable and chaotic regimes.
- Growing forever does not make sense: the Krylov-Lanczos space does not represent the system Hilbert space systematically.

Arnoldi iteration

- For non-Hermitian case, the natural choice for constructing Krylov basis is actually the Krylov-Arnoldi iteration.
- Results in a matrix form of the Lindbladian that is upper Hessenberg

$$\mathscr{L}^{(o)} = \begin{pmatrix} h_{0,0} & h_{0,1} & h_{0,2} & \cdots & h_{0,n} \\ h_{1,0} & h_{1,1} & h_{1,2} & \cdots & h_{1,n} \\ 0 & h_{2,1} & h_{2,2} & h_{2,3} & \cdots \\ \cdots & \cdots & h_{3,2} & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & h_{n-1,n} \\ 0 & 0 & \cdots & h_{n,n-1} & h_{n,n} \end{pmatrix}.$$

- In this case, the Lindbladian acting on a basis vector produces, not just contributions from the previous and next basis, but all existing basis vectors upto the end of Krylov basis.

- While applying the algorithm, and constructing $|\mathcal{O}_{n+1})$ from $|\mathcal{O}_n)$, therefore, one needs to subtract contributions for all $|\mathcal{O}_m)$, with $m = 0,1,\cdots,n-1$.

- These overlaps with previous elements form the matrix elements $h_{m,n}$.

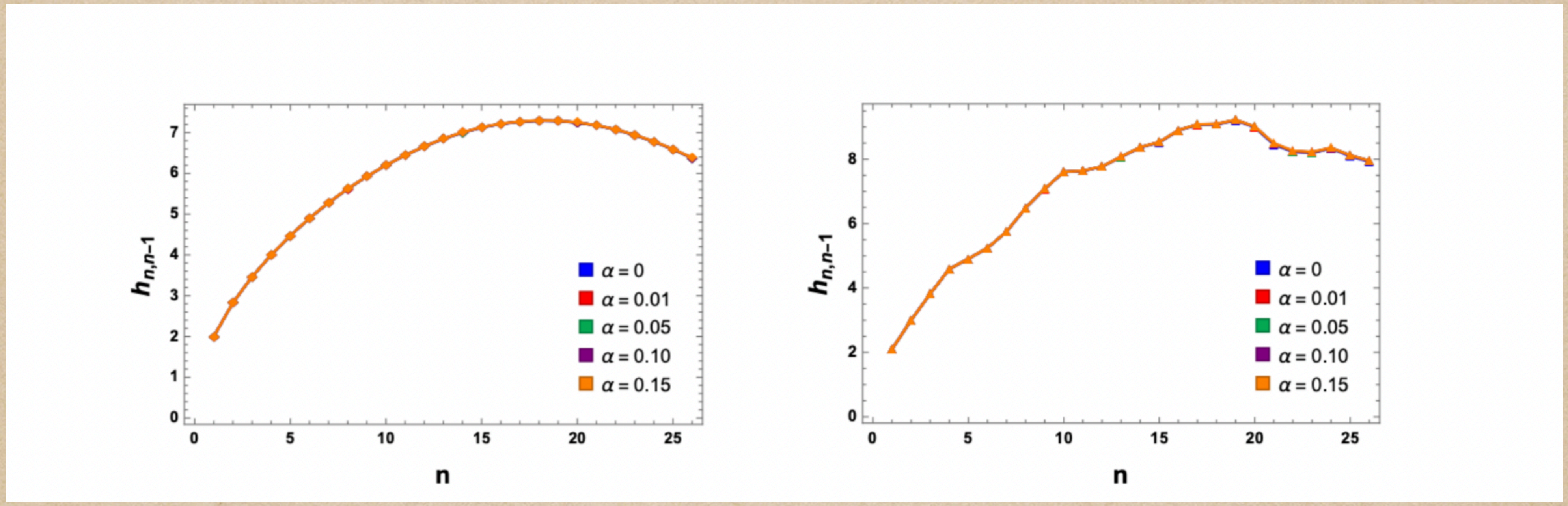
- Questions: 1) Do these coefficients have enough information about integrability? 2) Do these coefficients have info about non-hermiticity?

- Answers: Affirmative.

Results

- The coefficients $h_{n,n-1}$, (analogous to b_n's) always have information about the integrability.
- Mostly insensitive to non-hermiticity→ change is very small with increasing α (boundary coupling) and γ (bulk dephasing).
- $h_{n,n-1}$ are always real. So where is the information of nonhermiticity of the Lindbladian →diagonal elements $h_{n,n}$.
- $h_{n,n} = ia_n$ are fully imaginary, and are sensitive to α and γ, but are insensitive to integrability.

$h_{n,n-1}$ plots



h_{n,n-1}
n
α = 0
α = 0.01
α = 0.05
α = 0.10
α = 0.15
h_{n,n-1}
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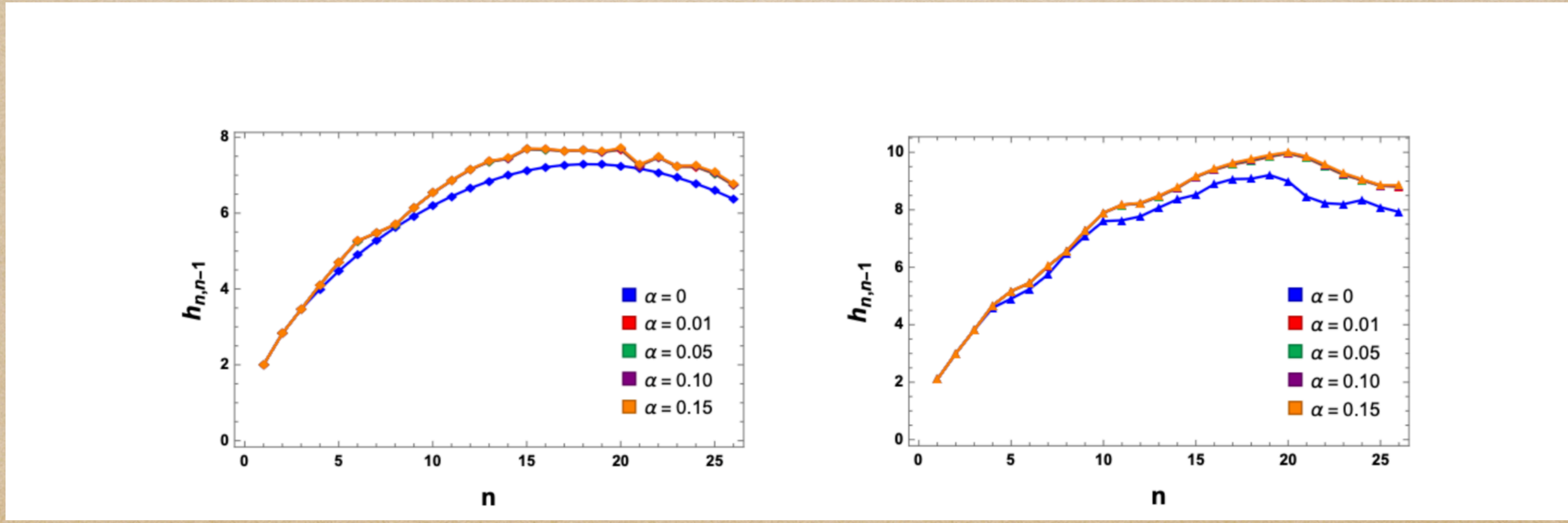
α = 0
α = 0.01
α = 0.05
α = 0.10
α = 0.15



h_{n,n-1}
n
α = 0
α = 0.01
α = 0.05
α = 0.10
α = 0.15
h_{n,n-1}
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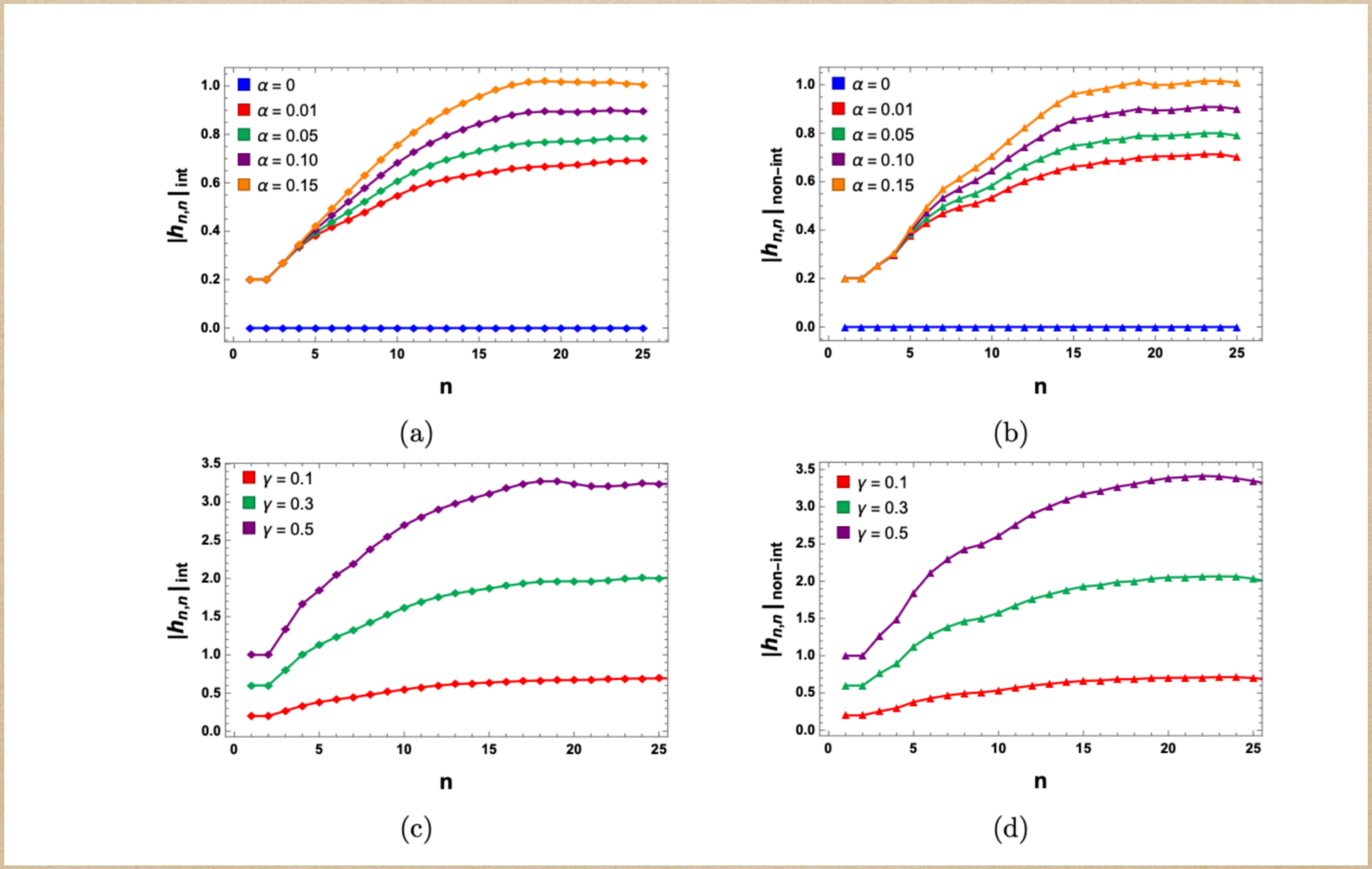
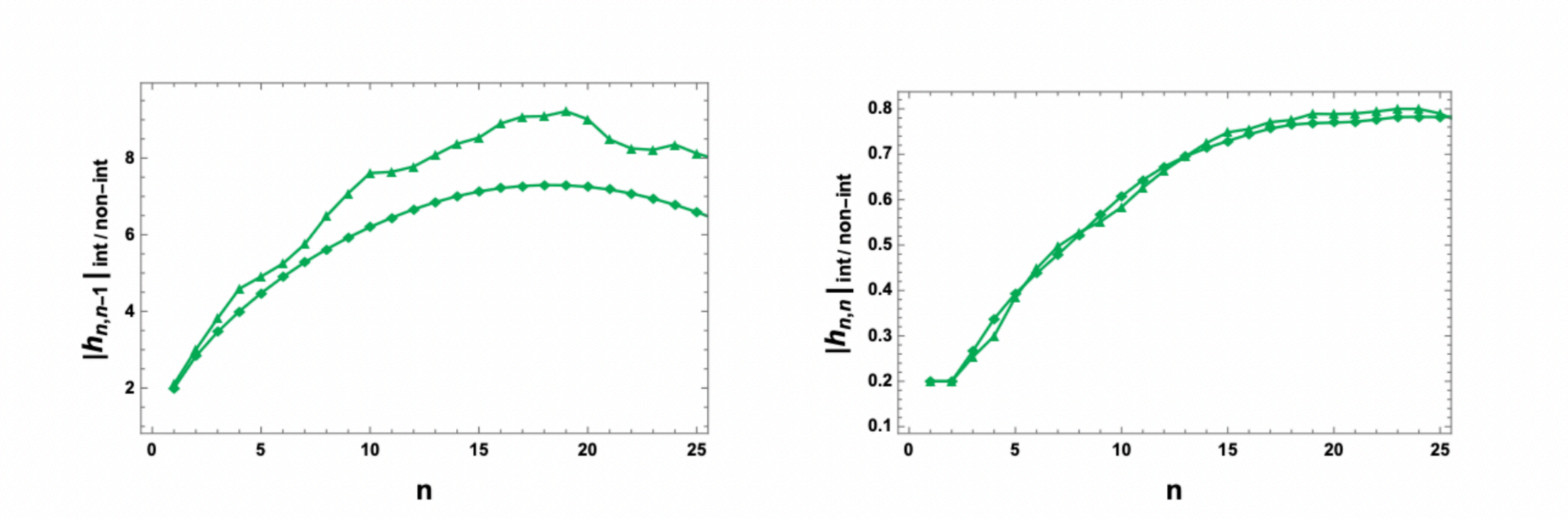
α = 0
α = 0.01
α = 0.05
α = 0.10
α = 0.15

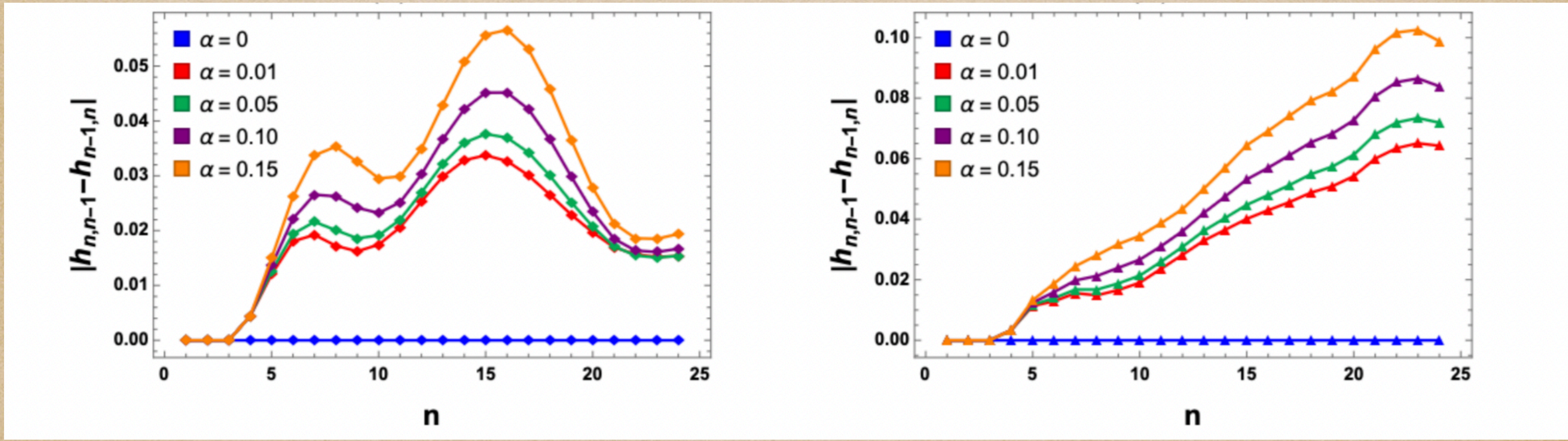
$h_{n,n}$ plots



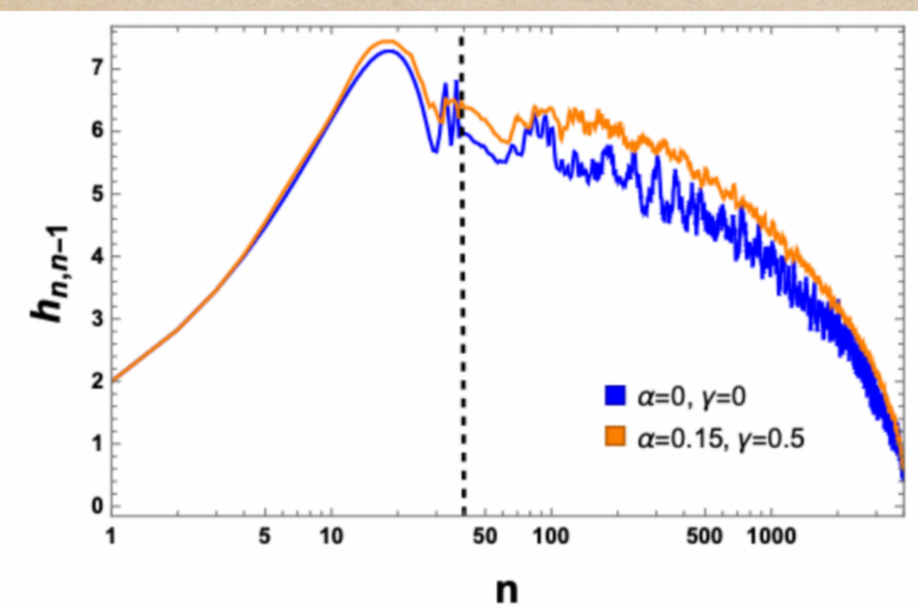
$h_{n,n}$ plots

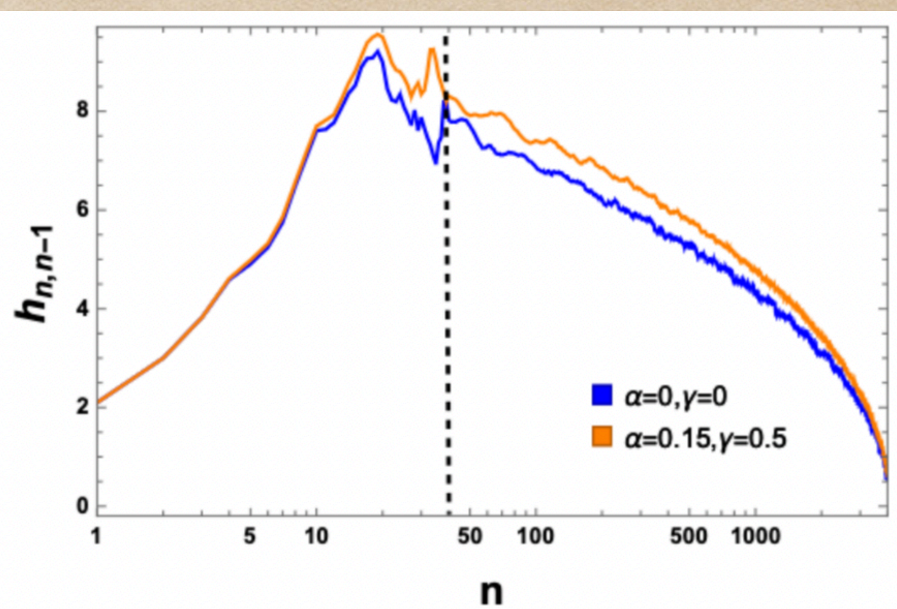


- Symmetrically placed partner coefficients of $h_{n,n-1}$ are the $h_{n-1,n}$ coefficients, that were equal for Lanczos are not equal anymore



- Finally, if we plot the $h_{n,n-1}$ Arnoldi coefficients for very large number of n, (we consider N=6, so $D = 2^6 = 64$ and $\mathcal{K} \leq 4^6 - 2^6 + 1 = 4033$), we find they go to zero, indicating a full exploration of the Krylov space. (For both integrable and chaotic)
- As expected, we find integrable coefficients showing more fluctuations later on.



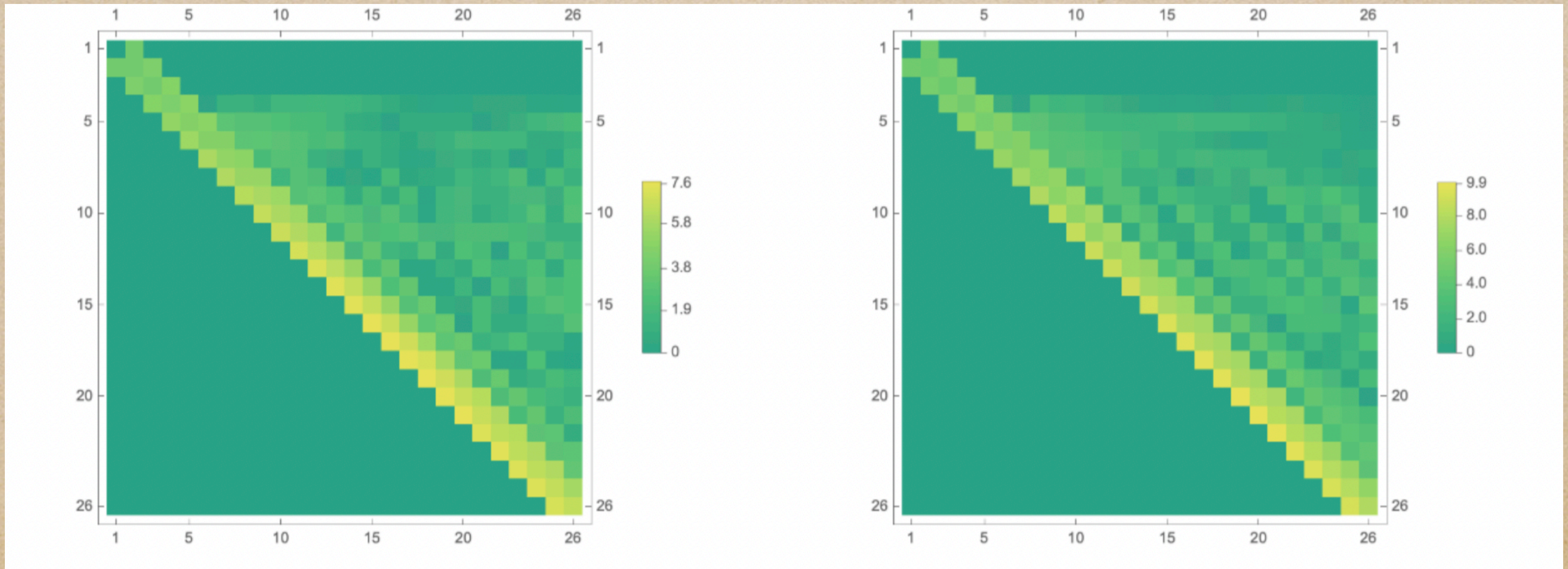


Conclusion

- Krylov-Lanczos iteration breaks down for open system non-unitary evolution.
- The forever growing Lanczos coefficients are not able to explain systematic exploration of system degrees of freedom and the corresponding Krylov basis.
- Krylov-Arnoldi seems to be the right procedure. Two different sets of coefficients capture the info about integrability and non-hermiticity separately.
- Systematic exploration of Krylov basis is regained.

Conclusion

- There are other nonzero matrix elements present, however they remain of the order of 10^{-2} and do not grow.
- These extra coefficients reflect that probability conservation is violated for open systems $\sum_{n=0}^{\mathcal{K}-1} |\phi_n(t)|^2 \neq 1$.
- This is expected since there is either a loss or a gain procedure for an open system due to its interaction.



- Arnoldi with non-hermitian couplings made zero boils down to Lanczos. So the difference between $h_{n,n-1}$ and $h_{n-1,n}$ seems to spread in the other small nonzero coefficients in the Arnoldi matrix.
- It would be interesting to study non-Hermitian Hamiltonians with unitary evolution (PT symmetric systems in PT unbroken phase).
- Computing complexity with full Arnoldi matrix becomes different. Following another way of biorthogonalizing the vector space gives one more hope of finding a matrix form similar to Lanczos, for which complexity computation should be doable.
- It would be interesting to study open QFTs and apply Arnoldi to see if the coefficients keep growing due to infinite degrees of freedom.

Thank you for your attention!