Non-Invertible Symmetries and Higher-Categories

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Global Symmetries of Quantum Field Theories

Why are we interested?

Global symmetries provide a powerful tool to probe strongly-coupled physical phenomena as they are RG flow invariant.

In the IR: Provide insight into the phase structure, confinement etc.

Typically useful in \( d \leq 4 \). [Aharony, Benini, LB, Cordova, Delmastro, DeWolfe, Gomis, Hidaka, Higginbotham, Hsin, Hubner, Eckhard, Gaiotto, Iqbal, Kapustin, Kim, Komargodski, Lam, Nitta, Ohmori, Poovuttikul, Razamat, Schafer-Nameki, Seiberg, Shao, Sharpe, Tachikawa, Thorngren, Willett, Yokokura, · · · ]

In the UV: Provide insight into the spectrum and properties of extended defects and local operators.

Typically useful in \( d \geq 4 \). [Cordova, Dumitrescu, Intriligator 2018] [LB 2021], [Lee, Ohmori, Tachikawa 2021], [LB, Bullimore, Ferrari, Schafer-Nameki 2022]

**Modern point of view:** Global symmetries correspond to topological defect operators in the theory.

**Standard global symmetry group** $G$ corresponds to topological codimension-one defects with $G$ fusion:

$$D_{d-1}^{(g)} \quad D_{d-1}^{(g')} = D_{d-1}^{(gg')}$$
$p$-form global symmetry group $G$ corresponds to topological codimension-$(p + 1)$ defects with $G$ fusion [Gaiotto, Kapustin, Seiberg, Willett 2014]:

\[
D^{(g)}_{d-p-1} + D^{(g')}_{d-p-1} = D^{(gg')}_{d-p-1}
\]
Non-invertible symmetries instead have a fusion ring:

\[ D_{d-p-1}^{(a)} \oplus D_{d-p-1}^{(b)} = \bigoplus_c D_{d-p-1}^{(c)} \]
Recent Activity in $d > 3$

Lots of activity within the last year:

[Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela 2021]
[Koide, Nagoya, Yamaguchi 2021]
[Kaidi, Ohmori, Zheng 2021]
[Choi, Cordova, Hsin, Lam, Shao 2021]
[Roumpedakis, Seifnashri, Shao 2022]
[LB, Bottini, Schafer-Nameki, Tiwari 2022]
[Choi, Cordova, Hsin, Lam, Shao 2022]
[Kaidi, Zafrir, Zheng 2022]
[Choi, Lam, Shao 2022]
[Cordova, Ohmori 2022]
[Antinucci, Galati, Rizi 2022]
[Aguilera Damia, Argurio, Garcia Valdecasas 2022]
The fusion rings are expected to be part of a fusion \((d - 1)\)-category:

Systematic study of higher-categorical nature of symmetries was initiated in [LB, Bottini, Schafer-Nameki, Tiwari 2022].
Fusion (and braided) higher-categories are hard to study:

1. Hard to provide a full set of axioms, i.e. all ‘irreducible’ consistency conditions following from topological moves. Solved for fusion 2-categories. \[\text{[Cui 2016], [Douglas, Reutter 2018]}\]

2. Hard to provide examples satisfying the above set of axioms.

**Proposal:** ‘Equivariantize’ higher-form symmetry categories

\[=\text{Gauge 0-form symmetry acting on higher-form symmetries}\]
Consider 3d pure gauge theory with gauge group Spin(4N).
1-form symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponding to topological lines forming a 1-category:

$$\mathcal{C}_{\text{Spin}(4N)}^{\text{ob}} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

0-form outer-automorphism (charge conjugation) symmetry:

$$D_1^{(S)} \longleftrightarrow D_1^{(C)}$$
Gauging outer-automorphism leads to $\text{Spin}(4N) \rtimes \mathbb{Z}_2 = \text{Pin}^+(4N)$ gauge theory.

$D_1^{(S)}$ and $D_1^{(C)}$ are not gauge invariant, but

$$D_1^{(SC)} := D_1^{(S)} \oplus D_1^{(C)}$$

is. Moreover, we get a $\mathbb{Z}_2$ Wilson line $D_1^{(-)}$. Thus in total, the new simple objects are

$$\mathcal{C}_{\text{Pin}^+(4N)}^{\text{ob}} = \left\{ D_1^{(id)}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V-)} \right\}$$
\[ D_1^{(S)} \leftrightarrow D_1^{(C)} \]

\[ \Rightarrow D_0^{(S)} + D_0^{(C)} \quad D_0^{(S)} - D_0^{(C)} \]

\[ D_1^{(SC)} \leftrightarrow D_1^{(SC)} \]
\[ \mathcal{D}_1^{(C)} \quad \mathcal{D}_1^{(V)} \quad \mathcal{D}_1^{(S)} \quad \mathcal{D}_1^{(V)} \]

\[ \mathcal{D}_0^{(1)} \quad \mathcal{D}_0^{(2)} \]

\[ \mathcal{D}_1^{(S)} \quad \mathcal{D}_1^{(C)} \]

\[ \mathcal{D}_1^{(SC)} \quad \mathcal{D}_1^{(SC)} \]

\[ \Rightarrow \quad \mathcal{D}_0^{(1)} + \mathcal{D}_0^{(2)} \quad \mathcal{D}_0^{(1)} - \mathcal{D}_0^{(2)} \]
Fusion rules:

\[ D_1^{(SC)} \otimes D_1^{(-)} = D_1^{(SC)} \]
\[ D_1^{(SC)} \otimes D_1^{(V)} = D_1^{(SC)} \]
\[ D_1^{(SC)} \otimes D_1^{(V-)} = D_1^{(SC)} \]
\[ D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(id)} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V-)} \]

Associator:

\[ C\text{Pin}^+(4N) = \text{Rep}(D_8) \]

**Nice physical interpretation:**

Gauging \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) 1-form symmetry of Spin(4N) leads to 3d theory with gauge group \( PSO(4N) \) which has \( (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = D_8 \) 0-form symmetry.

The \( \text{Pin}^+(4N) \) theory is obtained by gauging this \( D_8 \) symmetry.
Equivariantization of 1-category

[Drinfeld, Gelaki, Nikshych, Ostrik 2009], [Barkeshli, Bonderson, Cheng, Wang 2014]

Can be easily generalized to a finite group $G$ acting on a (multi-)tensor 1-category $\mathcal{C}$ (with suitable extra structures) describing symmetries of a $d$-dimensional theory.

After gauging $G$, we obtain the $G$-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$, which is another (multi-)tensor 1-category.

The objects of $\mathcal{C}_G$ are built by combining:

1. $G$-invariant objects of $\mathcal{C}$.
   
   Physically, these are gauge invariant topological line defects.

2. $G$ Wilson lines, which are objects of $\text{Rep}(G)$.

   Physically, these are 1d TQFTs with $G$ global symmetry.
$d$-dim theory with $G$ symmetry

$1d$ TQFT with $G$ symmetry

$G$ gauging

$d$-dim theory with $G$ gauged

Topological line defect
The morphisms of $C_G$ are built from:

1. Morphisms of $C$ organized according to $G$-reps.
   
   Physically, these are topological local operators attached to $G$
   Wilson lines.

2. Morphisms of $\text{Rep}(G)$.
   
   Physically, these are $G$-invariant topological interfaces between 1d
   TQFTs with $G$ symmetry.

\[
\begin{array}{c}
\text{d-dim theory with } G \text{ symmetry} \\
\begin{array}{c}
L_2 \\
L_1
\end{array}
\end{array}
\quad G \text{ gauging}
\quad
\begin{array}{c}
\text{d-dim theory with } G \text{ gauged} \\
\begin{array}{c}
L_2 \\
L_1
\end{array}
\end{array}
\]
Equivariantization of 2-category

[LB, Schafer-Nameki, Wu (soon)]

Similarly, consider a 2-category $\mathcal{C}$ describing symmetries of a $d$-dim theory with an action of $G$ on it.

After gauging $G$, we obtain the $G$-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$, which is another 2-category. See [Bernaschini, Galindo, Mombelli 2017] for prior work.

The objects of $\mathcal{C}_G$ are built by combining:

1. $G$-invariant objects of $\mathcal{C}$.
   
   Physically, these are gauge invariant topological surface defects.

2. Objects of $\text{2-Rep}(G) = \{\text{Category of modules of } \text{Vec}_G\}$.
   
   Physically, these are 2d TQFTs with $G$ global symmetry.
$d$-dim theory with $G$ symmetry

$G$ gauging

$2d$ TQFT with $G$ symmetry

Topological surface defect
The 1-morphisms of $C_G$ are built from:

1. 1-morphisms of $C$ organized according to how $G$ acts on them. 
   Physically, these are topological line defects attached to objects of
   $2\text{-Rep}(G)$. E.g. the $G$-action on line might carry a ‘t Hooft anomaly.

2. 1-morphisms of $2\text{-Rep}(G)$.
   Physically, these are $G$-symmetric topological interfaces between 2d
   TQFTs with $G$ symmetry.

\[
\begin{array}{ccc}
\text{d-dimensional theory with } G & \text{symmetry} & \text{d-dimensional theory with } G \text{ gauged} \\
\downarrow & G \text{ gauging} & \\
L & & L \\
\end{array}
\]

$\in 2\text{-Rep}(G)$
Non-invertible 2-category from invertible 2-category

Consider 4d pure gauge theory with gauge group $\text{Spin}(4N)$.

1-form symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponding to topological surfaces forming a 2-category:

$$
\mathcal{C}_{\text{Spin}(4N)}^{\text{ob}} = \left\{ D_{2}^{(\text{id})}, D_{2}^{(S)}, D_{2}^{(C)}, D_{2}^{(V)} \right\}
$$

$$
\mathcal{C}_{\text{Spin}(4N)}^{1\text{-endo}} = \left\{ D_{1}^{(\text{id})}, D_{1}^{(S)}, D_{1}^{(C)}, D_{1}^{(V)} \right\}
$$

0-form outer-automorphism (charge conjugation) symmetry:

$$
D_{i}^{(S)} \longleftrightarrow D_{i}^{(C)}
$$
Need to look for 2d TQFTs with $\mathbb{Z}_2$ symmetry: $\mathbb{Z}_2$ gauge theory is the only non-trivial choice.

$$C_{\text{Pin}^+(4N)}^{\text{ob}} = \left\{ D_2^{(\text{id})}, D_2^{(SC)}, D_2^{(V)}, D_2^{(\mathbb{Z}_2)}, D_2^{(V_{\mathbb{Z}_2})} \right\}$$

$D_2^{(\mathbb{Z}_2)}$ can also be recognized as condensation defect: Obtained by gauging dual $\mathbb{Z}_2$ 1-form symmetry on a surface. Studied recently in [Roumpedakis, Seifnashri, Shao 2022], [Choi, Cordova, Hsin, Lam, Shao 2022].

Unlike the line $D_1^{(-)}$, the surface $D_2^{(\mathbb{Z}_2)}$ is non-invertible:

$$D_2^{(\mathbb{Z}_2)} \otimes D_2^{(\mathbb{Z}_2)} \cong 2D_2^{(\mathbb{Z}_2)}$$

$D_2^{(\mathbb{Z}_2)}$ cannot be decoupled from the other surface defects because of the fusion

$$D_2^{(SC)} \otimes D_2^{(SC)} \cong D_2^{(\mathbb{Z}_2)} \oplus D_2^{(V_{\mathbb{Z}_2})}$$
Encoded in 2-category $C_G$ as the fact that there exists a 2-morphism from $D_1^{(SC)} \otimes D_1^{(SC)}$ to $D_1^{(-)}$. 
Equivariantization of 3-category?

Similarly we can consider defining the $G$-equivariantization $\mathcal{C}_G$ of a 3-category $\mathcal{C}$ with an action of $G$, which is another 3-category built by combining:

1. $G$-invariant objects and morphisms of $\mathcal{C}$.

2. Objects and morphisms of $3\text{-Rep}(G) := \{\text{Category of modules of } 2\text{-Vec}_G\}$.

However, the physical problem is apriori much larger. There are more $G$-symmetric 3d TQFTs than the ones captured by $3\text{-Rep}(G)$. The latter are those $G$-symmetric 3d TQFTs whose underlying 3d TQFTs do not carry topological order.
Other Examples of Higher-Categorical Symmetries

Just need a 0-form symmetry acting on higher-form symmetries.

Many interesting examples:

- In 5d, one can consider $\mathcal{N} = 2$ SYM theory with $\text{Spin}(4N)$ gauge group, and gauge outer-automorphism.
- In 6d, one can consider (absolute) $\mathcal{N} = (2,0)$ SCFT of type $SO(4N) \times SO(4N)$, and gauge exchange symmetry.
- In 4d, one can consider $\mathcal{N} = 3$ theories obtained by gauging $S$-duality of $\mathcal{N} = 4$ SYM theories. [LB, Buican, Radhakrishnan (soon)]
Conclusion

Beginning of a new symbiotic relationship between (non-topological) quantum field theory and higher-category theory:

- Physical properties of non-invertible symmetries in QFT are conveniently encoded in terms of higher-categories. Thus, the study of non-invertible symmetries in QFT is a natural place for applications of higher-category theory.

- On the other hand, intuitive ideas used in this physical context can spur new developments in higher-category theory. Moreover, the physical applications provide a motivation to explore higher-category theory further.
Future Directions

- Can also consider de-equivariantization/condensation = gauging of higher-form symmetries. This leads to webs of higher-categories connected to each other via equivariantization/de-equivariantization processes. [LB, Bottini, Schafer-Nameki, Tiwari (soon)]

- Action of higher-categorical symmetries on (non-topological) defect operators. [LB, Bullimore, Ferrari, Schafer-Nameki (in progress)]

- Much more to do! (Invitation)