Non-Invertible Symmetries and Higher-Categories

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Global Symmetries of Quantum Field Theories

Why are we interested?

Global symmetries provide a powerful tool to probe strongly-coupled physical phenomena as they are RG flow invariant.

In the IR: Provide insight into the phase structure, confinement etc.

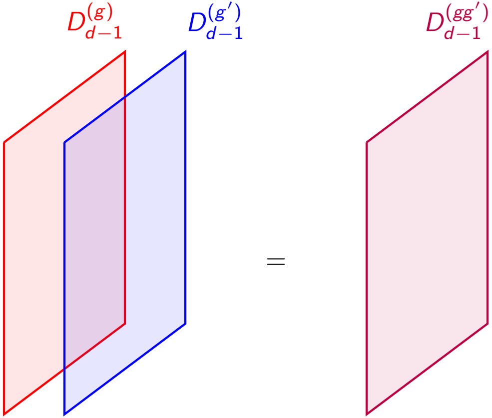
Typically useful in $d \leq 4$. [Aharony, Benini, LB, Cordova, Delmastro, DeWolfe, Gomis, Hidaka, Higginbotham, Hsin, Hubner, Eckhard, Gaiotto, Iqbal, Kapustin, Kim, Komargodski, Lam, Nitta, Ohmori, Poovuttikul, Razamat, Schafer-Nameki, Seiberg, Shao, Sharpe, Tachikawa, Thorngren, Willett, Yokokura, $\cdots$]

In the UV: Provide insight into the spectrum and properties of extended defects and local operators. [Cordova, Dumitrescu, Intriligator 2018] [LB 2021], [Lee, Ohmori, Tachikawa 2021], [LB, Bullimore, Ferrari, Schafer-Nameki 2022]

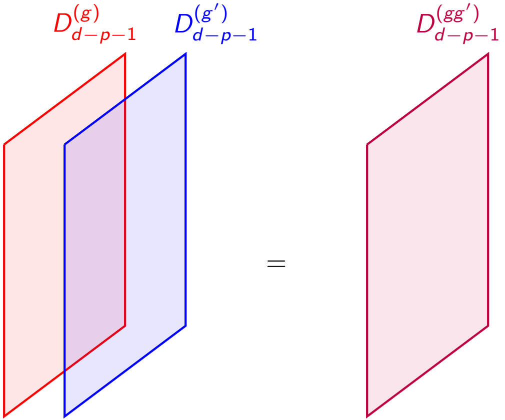
Typically useful in $d \geq 4$. [Albertini, Apruzzi, van Beest, LB, Bonetti, Closset, Cvetic, Del Zotto, Dierigl, Garcia Etxebarria, Giacomelli, Gould, Heckman, Heidenreich, Hubner, Hosseini, Lin, Meynet, Morrison, Moscrop, Oh, Regelado, Schafer-Nameki, Torres, Wang, Zhang, $\cdots$]

Modern point of view: Global symmetries correspond to topological defect operators in the theory.

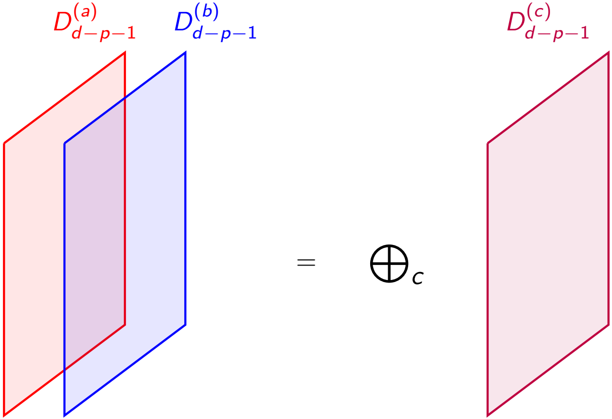
Standard global symmetry group G corresponds to topological codimension-one defects with G fusion:



p-form global symmetry group G corresponds to topological codimension-$(p+1)$ defects with G fusion [Gaiotto, Kapustin, Seiberg, Willett 2014]:



Non-invertible symmetries instead have a fusion ring:



Recent Activity in $d > 3$

Lots of activity within the last year:

[Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela 2021]

[Koide, Nagoya, Yamaguchi 2021]

[Kaidi, Ohmori, Zheng 2021]

[Choi, Cordova, Hsin, Lam, Shao 2021]

[Roumpedakis, Seifnashri, Shao 2022]

[LB, Bottini, Schafer-Nameki, Tiwari 2022]

[Choi, Cordova, Hsin, Lam, Shao 2022]

[Kaidi, Zafrir, Zheng 2022]

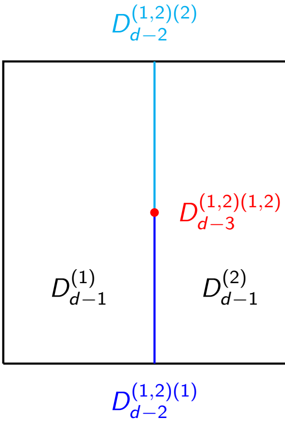
[Choi, Lam, Shao 2022]

[Cordova, Ohmori 2022]

[Antinucci, Galati, Rizi 2022]

[Aguilera Damia, Argurio, Garcia Valdecasas 2022]

The fusion rings are expected to be part of a fusion $(d-1)$-category:



Systematic study of higher-categorical nature of symmetries was initiated in [LB, Bottini, Schafer-Nameki, Tiwari 2022].

Hard Problem

Fusion (and braided) higher-categories are hard to study:

1. Hard to provide a full set of axioms, i.e. all ‘irreducible’ consistency conditions following from topological moves. Solved for fusion 2-categories. [Cui 2016], [Douglas, Reutter 2018]
2. Hard to provide examples satisfying the above set of axioms.

Proposal: ‘Equivariantize’ higher-form symmetry categories

$=$ Gauge 0-form symmetry acting on higher-form symmetries

Non-invertible 1-category from invertible 1-category

Consider 3d pure gauge theory with gauge group Spin($4N$).

1-form symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponding to topological lines forming a 1-category:

$$\mathcal{C}^{\text{ob}}_{\text{Spin}(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

0-form outer-automorphism (charge conjugation) symmetry:

$$D_1^{(S)} \longleftrightarrow D_1^{(C)}$$

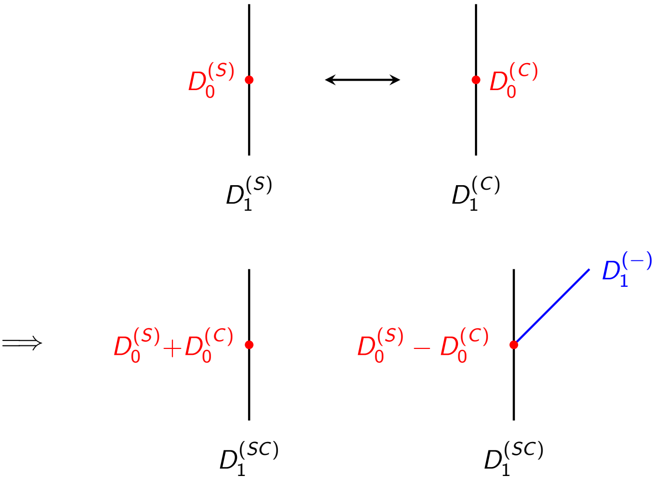
Gauging outer-automorphism leads to $\mathrm{Spin}(4N) \rtimes \mathbb{Z}_2 = \mathrm{Pin}^+(4N)$ gauge theory.

$D_1^{(S)}$ and $D_1^{(C)}$ are not gauge invariant, but

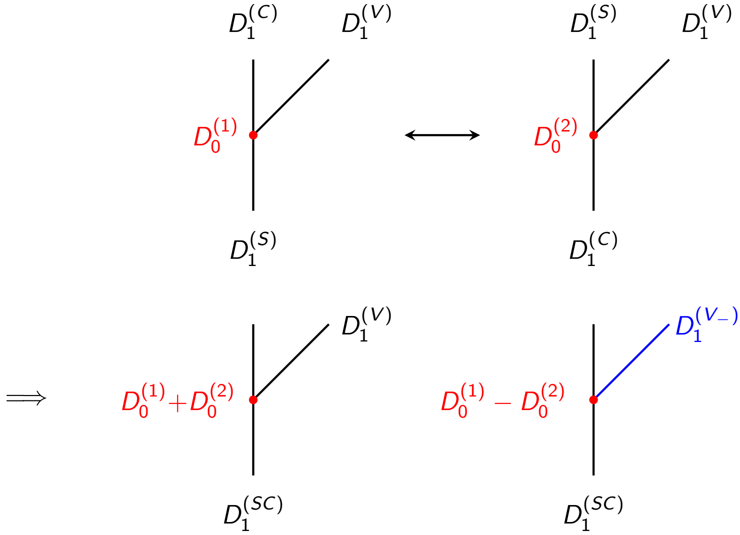
$$D_1^{(SC)} := D_1^{(S)} \oplus D_1^{(C)}$$

is. Moreover, we get a $\mathbb{Z}_2$ Wilson line $D_1^{(-)}$. Thus in total, the new simple objects are

$$\mathcal{C}^{\mathrm{ob}}_{\mathrm{Pin}^+(4N)} = \left\{ D_1^{(\mathrm{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)} \right\}$$



$D_0^{(S)}$
$D_0^{(C)}$
$D_1^{(S)}$
$D_1^{(C)}$
$\Longrightarrow$
$D_1^{(-)}$
$D_0^{(S)}+D_0^{(C)}$
$D_0^{(S)}-D_0^{(C)}$
$D_1^{(SC)}$
$D_1^{(SC)}$



$D_1^{(C)}$
$D_1^{(V)}$
$D_0^{(1)}$
$D_1^{(S)}$
$D_1^{(S)}$
$D_1^{(V)}$
$D_0^{(2)}$
$D_1^{(C)}$
$\Longrightarrow$
$D_1^{(V)}$
$D_0^{(1)}+D_0^{(2)}$
$D_1^{(SC)}$
$D_1^{(V_-)}$
$D_0^{(1)} - D_0^{(2)}$
$D_1^{(SC)}$

Fusion rules:

$$D_1^{(SC)} \otimes D_1^{(-)} = D_1^{(SC)}$$
$$D_1^{(SC)} \otimes D_1^{(V)} = D_1^{(SC)}$$
$$D_1^{(SC)} \otimes D_1^{(V_-)} = D_1^{(SC)}$$
$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\mathrm{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

Associator:

$$\mathcal{C}_{\mathrm{Pin}^+(4N)} = \mathrm{Rep}(D_8)$$

Nice physical interpretation:

Gauging $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry of Spin($4N$) leads to 3d theory with gauge group $PSO(4N)$ which has $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = D_8$ 0-form symmetry. The $\mathrm{Pin}^+(4N)$ theory is obtained by gauging this D_8 symmetry.

Equivariantization of 1-category

[Drinfeld, Gelaki, Nikshych, Ostrik 2009], [Barkeshli, Bonderson, Cheng, Wang 2014]

Can be easily generalized to a finite group G acting on a (multi-)tensor 1-category $\mathcal{C}$ (with suitable extra structures) describing symmetries of a d-dimensional theory.

After gauging G, we obtain the G-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$, which is another (multi-)tensor 1-category.

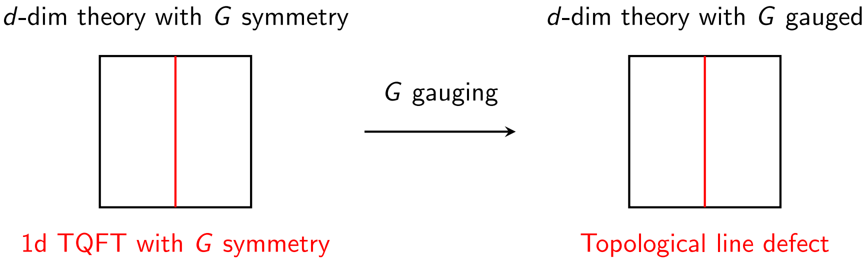
The objects of $\mathcal{C}_G$ are built by combining:

1. G-invariant objects of $\mathcal{C}$.

   Physically, these are gauge invariant topological line defects.

2. G Wilson lines, which are objects of Rep(G).

   Physically, these are 1d TQFTs with G global symmetry.


d-dim theory with G symmetry
d-dim theory with G gauged
G gauging
1d TQFT with G symmetry
Topological line defect

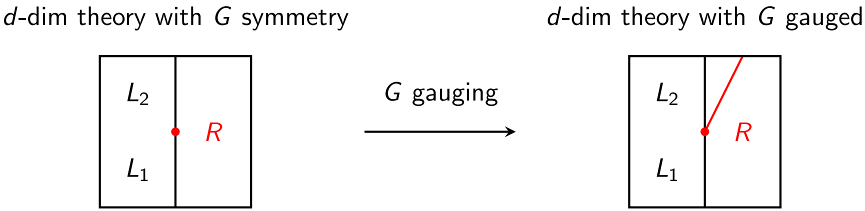
The morphisms of $\mathcal{C}_G$ are built from:

1. Morphisms of $\mathcal{C}$ organized according to G-reps.

   Physically, these are topological local operators attached to G Wilson lines.

2. Morphisms of Rep(G).

   Physically, these are G-invariant topological interfaces between 1d TQFTs with G symmetry.



Equivariantization of 2-category

[LB, Schafer-Nameki, Wu (soon)]

Similarly, consider a 2-category $\mathcal{C}$ describing symmetries of a d-dim theory with an action of G on it.

After gauging G, we obtain the G-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$, which is another 2-category. See [Bernaschini, Galindo, Mombelli 2017] for prior work.

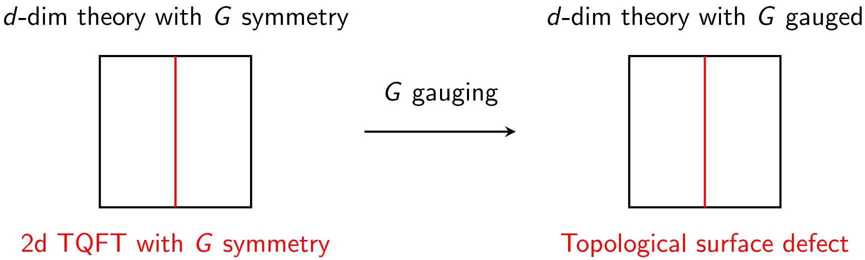
The objects of $\mathcal{C}_G$ are built by combining:

1. G-invariant objects of $\mathcal{C}$.

   Physically, these are gauge invariant topological surface defects.

2. Objects of 2-Rep(G)=$\{$Category of modules of Vec$_G\}$.

   Physically, these are 2d TQFTs with G global symmetry.


d-dim theory with G symmetry
G gauging
d-dim theory with G gauged
2d TQFT with G symmetry
Topological surface defect

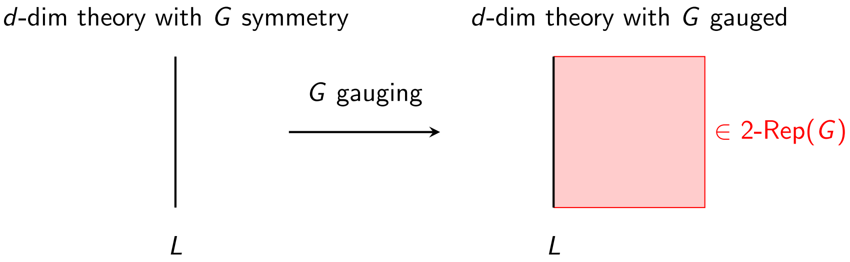
The 1-morphisms of $\mathcal{C}_G$ are built from:

1. 1-morphisms of $\mathcal{C}$ organized according to how G acts on them.

   Physically, these are topological line defects attached to objects of 2-Rep(G). E.g. the G-action on line might carry a ‘t Hooft anomaly.

2. 1-morphisms of 2-Rep(G).

   Physically, these are G-symmetric topological interfaces between 2d TQFTs with G symmetry.



Non-invertible 2-category from invertible 2-category

Consider 4d pure gauge theory with gauge group Spin$(4N)$.

1-form symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponding to topological surfaces forming a 2-category:

$$\mathcal{C}^{\text{ob}}_{\text{Spin}(4N)} = \left\{ D_2^{(\text{id})}, D_2^{(S)}, D_2^{(C)}, D_2^{(V)} \right\}$$

$$\mathcal{C}^{\text{1-endo}}_{\text{Spin}(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

0-form outer-automorphism (charge conjugation) symmetry:

$$D_i^{(S)} \longleftrightarrow D_i^{(C)}$$

Need to look for 2d TQFTs with $\mathbb{Z}_2$ symmetry: $\mathbb{Z}_2$ gauge theory is the only non-trivial choice.

$$\mathcal{C}^{\text{ob}}_{\text{Pin}^+(4N)} = \left\{ D_2^{(\text{id})}, D_2^{(SC)}, D_2^{(V)}, D_2^{(\mathbb{Z}_2)}, D_2^{(V_{\mathbb{Z}_2})} \right\}$$

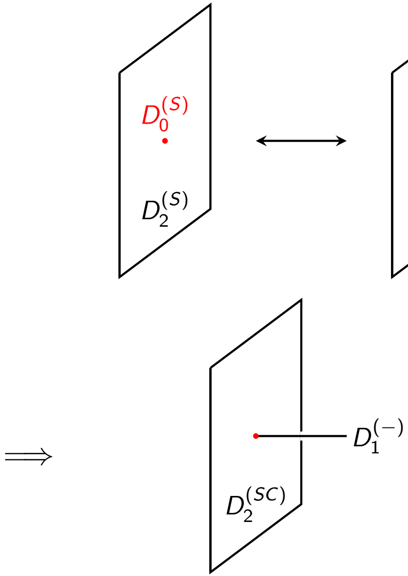
$D_2^{(\mathbb{Z}_2)}$ can also be recognized as condensation defect: Obtained by gauging dual $\mathbb{Z}_2$ 1-form symmetry on a surface. Studied recently in [Roumpedakis, Seifnashri, Shao 2022], [Choi, Cordova, Hsin, Lam, Shao 2022].

Unlike the line $D_1^{(-)}$, the surface $D_2^{(\mathbb{Z}_2)}$ is non-invertible:

$$D_2^{(\mathbb{Z}_2)} \otimes D_2^{(\mathbb{Z}_2)} \cong 2 D_2^{(\mathbb{Z}_2)}$$

$D_2^{(\mathbb{Z}_2)}$ cannot be decoupled from the other surface defects because of the fusion

$$D_2^{(SC)} \otimes D_2^{(SC)} \cong D_2^{(\mathbb{Z}_2)} \oplus D_2^{(V_{\mathbb{Z}_2})}$$



Encoded in 2-category $\mathcal{C}_G$ as the fact that there exists a 2-morphism from $D_1^{(SC)} \otimes D_1^{(SC)}$ to $D_1^{(-)}$.

Equivariantization of 3-category?

Similarly we can consider defining the G-equivariantization $\mathcal{C}_G$ of a 3-category $\mathcal{C}$ with an action of G, which is another 3-category built by combining:

1. G-invariant objects and morphisms of $\mathcal{C}$.
2. Objects and morphisms of 3-Rep(G):={Category of modules of 2-Vec$_G$}.

However, the physical problem is apriori much larger. There are more G-symmetric 3d TQFTs than the ones captured by 3-Rep(G). The latter are those G-symmetric 3d TQFTs whose underlying 3d TQFTs do not carry topological order.

Other Examples of Higher-Categorical Symmetries

Just need a 0-form symmetry acting on higher-form symmetries.

Many interesting examples:

- In 5d, one can consider $\mathcal{N} = 2$ SYM theory with Spin($4N$) gauge group, and gauge outer-automorphism.
- In 6d, one can consider (absolute) $\mathcal{N} = (2, 0)$ SCFT of type $SO(4N) \times SO(4N)$, and gauge exchange symmetry.
- In 4d, one can consider $\mathcal{N} = 3$ theories obtained by gauging S-duality of $\mathcal{N} = 4$ SYM theories. [LB, Buican, Radhakrishnan (soon)]

Conclusion

Beginning of a new symbiotic relationship between (non-topological) quantum field theory and higher-category theory:

- Physical properties of non-invertible symmetries in QFT are conveniently encoded in terms of higher-categories. Thus, the study of non-invertible symmetries in QFT is a natural place for applications of higher-category theory.
- On the other hand, intuitive ideas used in this physical context can spur new developments in higher-category theory. Moreover, the physical applications provide a motivation to explore higher-category theory further.

Future Directions

- Can also consider de-equivariantization/condensation = gauging of higher-form symmetries. This leads to webs of higher-categories connected to each other via equivariantization/de-equivariantization processes. [LB, Bottini, Schafer-Nameki, Tiwari (soon)]
- Action of higher-categorical symmetries on (non-topological) defect operators. [LB, Bullimore, Ferrari, Schafer-Nameki (in progress)]
- Much more to do! (Invitation)