On topology change in Poisson-Lie T-duality

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I will mostly talk about [arXiv:2110.08179] [ASA, Chris Blair, Dan Thompson]. It's about **changes in the global bundle topology** (e.g. trivial to nontrivial) under a nonabelian analogue of T-duality — Poisson-Lie duality.

The result, briefly:

A spacetime which is a principal *bibundle* with Poisson-Lie group fibres is T-dual to another bibundle whose structure group is the PL dual group.

Nonabelian fibres/'isometries': e.g. $S^3 \cong \mathrm{SU}(2)$-bundles dual to $\mathbb{R}^3 \cong \mathrm{SB}(2)$-bundles. (Whereas in T-duality it's just torus bundles.)

Generalisation of topological T-duality to a nonabelian setting, via duality of QP mfolds/Courant/symplectic L_∞-algebroids $\implies$ **duality of sigma-models**.

Recall T-duality: strings on $B \times S^1_R$ dual to strings on $B \times S^1_{\alpha'/R}$ $\implies$ **change in geometry** (lengths).

How to see T-duality? [Roček Verlinde 91]: if d $U(1)$ isometries, introduce a **doubled** sigma model with d extra scalars; then gauge d currents $\implies$ equivalence of CFTs.

Schematically, for a single isometry:

$$B \times S^1_R \times S^1_{\alpha'/R}$$

gauging a current ↙ ↘ gauging another current

$$B \times S^1_R \qquad\qquad B \times S^1_{\alpha'/R}$$

The logic of this **correspondence** diagram is how various dualities work, including e.g. bosonisation. (For a 2D Dirac fermion, gauge the vector symmetry $\psi \to \exp(i\theta(x))\psi$ with a flat connection to get the top node/"*correspondence space*".)

The correspondence picture is abstract. Duality in phase space instead: **Hamiltonian formulation of Polyakov(–Howe-Tucker) string action** (zero B-field)

$$S[X,P;e,u]=\int dt\oint d\sigma\;\dot{X}^{\mu}P_{\mu}-e\Big(g^{-1}(P^2)+g(\partial_\sigma X^2)\Big)-u(\partial_\sigma X^{\mu}P_{\mu})\,.$$

$O(d,d)$ vector $\mathbb{Z}^M$ transforms nicely under T-duality:

$$\mathbb{Z}^M:=(\partial_\sigma X^{\mu},P_{\mu})\quad\to\quad \mathbb{O}^M{}_N\mathbb{Z}^N=:\widetilde{\mathbb{Z}}^M$$

for $\mathbb{O}\in O(d,d)$. For $\mathbb{O}$ that dualises all coordinates, a "canonical transformation" (lagrangian correspondence) [Álvarez Álvarez-Gaumé Lozano 94]

$$P=\partial_\sigma\widetilde{X}\,,\qquad \widetilde{P}=\partial_\sigma X\,,$$

which can be seen "passively" as acting on $\mathcal{H}=\begin{pmatrix}g^{-1}&0\\0&g\end{pmatrix}\in O(d,d)/O(d)\times O(d)$.

This canonical picture is good enough at least for cylindrical strings.

T-duality with spectators & topological T-duality

Instead of $M = B \times T^d$, could dualise a non-trivial torus bundle $T^d \hookrightarrow M \twoheadrightarrow B$.

New phenomenon: change in topology [Álvarez Álvarez-Gaumé Barbón Lozano 93]

$$S^3 \quad \text{without} \quad H\text{-flux} \qquad \xrightarrow{\text{T-dual to}} \qquad S^2 \times S^1 \quad \textbf{with} \quad H\text{-flux.}$$

Here S^3 is seen as a bundle $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$ (Hopf fibration) of Chern class 1.
Represent this via curvature F of a $U(1)$ connection (KK photon). **Dual H-flux**

$$\widetilde{H} = F \wedge \widetilde{A}, \qquad \widetilde{A} \propto \mathrm{vol}_{S^1}\,. \quad (\widetilde{A} \text{ the dual KK photon})$$

Original paper is a formidable patch-wise calculation...

[Bouwknegt Evslin Mathai 03] treat this topology change generally — **topological T-duality**:

- Principal S^1-bundle M with H-flux H and Chern class represented by 2-form F.
- T-duality proposed to swap F with the 2-form $\widetilde{F} = \int_{S^1} H$ (“momentum $\leftrightarrow$ winding”)
- Class $[\widetilde{F}]$ determines an S^1 bundle $\widetilde{M}$ (by algebraic topology) — **the dual space.**

$$\begin{array}{ccccc} & & \mathbb{M} \equiv M \times_B \widetilde{M} & & \\ & \swarrow^{p} & & \searrow^{\tilde{p}} & \\ M & & & & \widetilde{M} \\ & \searrow & & \swarrow & \\ & & B & & \end{array}$$

‘correspondence’/‘doubled’ space (T^2-bundle)

(S^1-bundles)

base/‘spectators’

Dual H flux determined via

$$H - \widetilde{H} = d(A \wedge \widetilde{A}) \qquad \text{on } M \times_B \widetilde{M}\,.$$

Crucial point

Original and dual H-fluxes are in the same cohomology class (on $\mathbb{M}$)!

Poisson-Lie T-duality [Klimčík Ševera 95] replaces circles/tori with **nonabelian groups.**

Naturally understood in the doubled picture. The fibre is a **Drinfeld double** Lie group:

Definition (Drinfeld double)

$\mathbb{D}$ is a Drinfeld double $\iff$ Lie algebra $\mathfrak{d} = \mathfrak{g} + \widetilde{\mathfrak{g}}$ is a *Manin triple*: has inner product η, so that η restricted to $\mathfrak{g}, \widetilde{\mathfrak{g}}$ is zero. (η is also nondegenerate and has signature $\mathrm{O}(d,d)$.)

Structure constants on $\mathfrak{d}$ totally determined by structure consts. $f, \tilde{f}$ on $\mathfrak{g}, \tilde{\mathfrak{g}}$:

$$[T_{\mathfrak{g}}, T_{\mathfrak{g}}] = fT_{\mathfrak{g}}\,, \quad [\tilde{T}_{\tilde{\mathfrak{g}}}, \tilde{T}_{\tilde{\mathfrak{g}}}] = \tilde{f}\,\tilde{T}_{\tilde{\mathfrak{g}}}\,, \qquad [T_{\mathfrak{g}}, \tilde{T}_{\tilde{\mathfrak{g}}}] = f\,\tilde{T}_{\tilde{\mathfrak{g}}} - \tilde{f}T_{\mathfrak{g}}\,,$$

PL dual subalgebras $\mathfrak{g}$ **and** $\tilde{\mathfrak{g}}$ **do not commute!**
I will assume that $\mathbb{D}$ is globally a (non-direct) product, so every $\mathbb{g} \in \mathbb{D}$ factorises:

$$\mathbb{g} = \tilde{g}g\,, \qquad g \in G, \quad \tilde{g} \in \widetilde{G}\,.$$

"Nonabelian factorisation" into d-dim Lie subgroups $G, \tilde{G}$ with Lie algebras $\mathfrak{g}, \widetilde{\mathfrak{g}}$

Example (The abelian double)

$\mathbb{D} = T^{2d}$, $G = T^d$, $\widetilde{G} = T^d$, relevant for usual T-duality.

Sigma models with targets $G, \widetilde{G}$ that fit into a Drinfeld double $\mathbb{D}$ can be PLT-dual.

Such G is a **Poisson-Lie group** for Poisson bivector $\Pi^{ab} = -\Pi^{ba}$. Comes from the **adjoint action** of G on the 'dual' Lie algebra $\tilde{\mathfrak{g}}$ (hence vanishes in the abelian double)

$$\widetilde{T}^a \to g\widetilde{T}^a g^{-1}\,, \qquad \widetilde{T}^a \in \tilde{\mathfrak{g}}\,, \quad g \in G.$$

The PL sigma model with target G is then (in lightcone coordinates)

$$S[g] = \int [\Pi + M_0]^{-1}_{ab} L^a_+ L^b_-\,, \qquad L_\pm = g^{-1}\partial_\pm g\,,$$

for M_0 constant $d \times d$ invertible matrix.

Amazingly two such sigma models are related by a "canonical transformation" [Sfetsos 97]

$$P = \widetilde{\Pi}\widetilde{P} + \tilde{L}_\sigma\,, \qquad \widetilde{P} = \Pi P + L_\sigma\,, \qquad (L_\sigma = g^{-1}\partial_\sigma g\,, \quad \tilde{L}_\sigma = \tilde{g}^{-1}\partial_\sigma \tilde{g}\,.)$$

For $\mathbb{D} = T^{2d}$, $\Pi = \widetilde{\Pi} = \tilde{f} = \cdots = 0$, $L_\sigma = \partial_\sigma X$, and we recover abelian T-duality.

PL duality obviously allows **topology change** when the groups are different manifolds:

Example

$$\mathbb{D} = \mathrm{SL}(2;\mathbb{C}),\ G = \mathrm{SU}(2) \cong S^3, \text{ and } \widetilde{G} = \left\{ \begin{pmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{pmatrix} \middle| \lambda > 0\,, z \in \mathbb{C} \right\};$$

more generaly $\mathbb{D} = G_{\mathbb{C}}$ for compact simple G is an example.

What about changes in the global fibration structure like in (topological) T-duality?

We did PL duality **with spectators** between e.g. trivial and non-trivial bundles. On top of PL group/Drinfeld double structure on fibres, we invoked **bibundle structure**.

Change of topology:	fibre	global bundle
Topological T–duality	NO	YES
Poisson-Lie duality	YES	NO
PL ‘bibundle’ duality	**YES**	**YES**

Top-down perspective: starting point is a principal **left** $\mathbb{D}$ bundle $\mathbb{M}$ with connection $\mathbb{A}$

$$\mathbb{D} \hookrightarrow \mathbb{M} \overset{\pi}{\twoheadrightarrow} B\,, \qquad \mathbb{A} = d\mathrm{g}\mathrm{g}^{-1} + \mathrm{g}\underline{\mathbb{A}}\mathrm{g}^{-1}$$

and H-flux involving a Chern-Simons term for $\mathbb{A}$ and a basic form h on B:

$$\mathbb{H} = \eta(\mathbb{A}\mathrm{d}\mathbb{A} + \mathbb{A}^3) + \pi^\star h\,, \qquad \mathrm{d}\mathbb{H} = 0 \iff \mathrm{d}h = \eta(\mathbb{F}\wedge\mathbb{F})\,.$$

Quotients $\mathbb{M}/\widetilde{G}$ and $\mathbb{M}/G$ should be PLT-dual, since $\mathbb{D}/\widetilde{G} = G$, $\mathbb{D}/G = \widetilde{G}$.

H-flux?

We expect (from abelian duality) H-flux on $\mathbb{M}$ is cohomologous to $\mathbb{M}/\tilde{G}$...

Indeed: for $\mathbb{D} = T^{2d}$, split $\mathbb{A} = A + \widetilde{A}$, so the H flux on $\mathbb{M}$ becomes

$$\eta(\mathbb{A}\mathrm{d}\mathbb{A}) = \widetilde{A}\mathrm{d}A + A\mathrm{d}\widetilde{A} = \mathrm{d}(A\widetilde{A}) + 2Ad\widetilde{A} \sim 2A\widetilde{F}$$

$A\widetilde{F}$ is $\widetilde{G} = T^d$-invariant and horizontal $\implies$ $A\wedge\widetilde{F}$ **descends** to $\mathbb{M}/\widetilde{G}$.

Essential nonabelian difficulty

For nonabelian $\mathbb{D}$, the factorisation $\mathbb{D} \ni \mathrm{g} = \tilde{g}g$ is noncommutative. So **left** $\tilde{G}$ cosets do not inherit the obvious **left** G action, even though $\mathbb{D}/\tilde{G} \cong G$.

We take a leap of faith and demand a **right** action of $\mathbb{D}$ on $\mathbb{M}$ alongside the **left** one:

Definition (Principal Bibundle [Breen 90, Aschieri Cantini Jurčo 03, Murray Roberts Stevenson 12])

A principal bundle $G \hookrightarrow M \twoheadrightarrow B$ for a left G action $\triangleright$ on M is a **principal bibundle** if it is a bundle for a right G action $\triangleleft$, and **both actions commute and have the same fibres**.

Example (Abelian bibundles)

All principal bundles with abelian structure group are bibundles.

Can understand instead as a **left** bundle equipped with a **structure map** $a : M \to \mathrm{Aut}(G)$ that writes the right action on some point $m \in M$ as a left one:

$$m \triangleleft g = a[m](g) \triangleright m$$

a is equivariant — morally, a *generalisation of the adjoint action $G \to G$ to a G-bundle*.

$a(F)$ is gauge-invariant even if F is a nonabelian field strength...

If $\mathbb{M}$ were a $\tilde{G}$-bibundle, we could write $A \wedge \tilde{\mathrm{a}}(\tilde{F})$ for the reduced H-flux (on $\mathbb{M}/\tilde{G}$).
Topological conditions?

Definition (Topological factorisation condition)

Structure group of the principal $\mathbb{D}$ bundle $\mathbb{M}$ reduces to $Z(G) \cap Z(\mathbb{D}) \times Z(\widetilde{G}) \cap Z(\mathbb{D})$.

Then $\mathbb{M}$ has $G, \tilde{G}, \mathbb{D}$-**bibundle structures**! The structure map $\mathbbm{a} : \mathbb{M} \to \mathrm{Aut}(\mathbb{D})$ factorises as $\mathbbm{a} = a\widetilde{a}$ — compare $\mathrm{g} = \tilde{g}g \implies \mathrm{Ad}(\mathrm{g}) = \mathrm{Ad}(\tilde{g})\mathrm{Ad}(g)$.

Example

$\mathbb{D} = \mathrm{SL}(2;\mathbb{C})$, $G = \mathrm{SU}(2) \cong S^3$, and $\widetilde{G} = \left\{ \begin{pmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{pmatrix} \middle| \lambda > 0 \,, z \in \mathbb{C} \right\}$. $Z(G) = \mathbb{Z}_2$,

$Z(\widetilde{G}) = 1 \implies$ **non-trivial** $\mathrm{SU}(2)$-bundles dualise to trivial $\widetilde{G}$ ones!

What do we mean by duality in this context?

- **Worldsheet**: 'canonical transfs'/symplectic reductions in string phase space
- **Target-space**: *idem* in certain **symplectic L_2-algebroids $\cong$ QP manifolds**

$\mathbb{Z}$-graded manifolds $\mathcal{M}$ (contrast 'super' $= \mathbb{Z}_2$-graded) with symPlectic form and compatible odd ('fermionic') vector field Q with $Q^2 = 0$.

Example

BV formalism: ∞-dim QP mfold (with symplectic form of degree -1).

We just need $\mathcal{M}_M = T^\star[2]T[1]M$. Darboux coordinates:

coord	x^μ	ψ^μ	χ_μ	p_μ
deg	0	1	1	2

$$\{x^\mu, p_\nu\} = \{\psi^\mu, \chi_\nu\} = \delta^\mu_\nu\,.$$

and Q-structure

$$Q = \mathrm{d}_{\text{e Rham}} + \{H(x)_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho, \bullet\}\,, \qquad Q^2 = 0 \iff \mathrm{d}H = 0\,,$$

for $H \in \Omega^3(M)$ the H-flux. These are **classified by the de Rham class of H.**

Our result: *dg-symplectic reduction* given a $\mathbb{D}$ bibundle as above.

For $\mathbb{M}$ to $\mathbb{M}/\widetilde{G}$ reduction, the $\mathbb{D}$-connection $\mathbb{A}$ on $\mathbb{M}$ produces

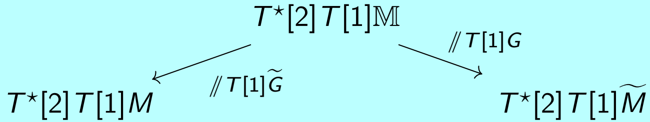
- an honest connection $\widetilde{A}$ for $\widetilde{G}$
- a G-connection-like object r^∇ ($\mathfrak{g}$-valued 1-form on $\mathbb{M}/\widetilde{G}$).

We found a formula for the H-flux (which is also the one for dual H-flux if we trade $G \leftrightarrow \widetilde{G}$)

$$H = r^\nabla \wedge r^\nabla \wedge \underline{\widetilde{A}} + r^\nabla \wedge \widetilde{a}(\widetilde{F}) + (\text{spectator flux})\,.$$

r^∇ is an honest connection whenever $\widetilde{G}$ is abelian!

Diagram of dg-symplectic reductions



whence a *lagr. correspondence* between $\mathcal{M}_M = T^\star[2]\,T[1]M$ and $\mathcal{M}_{\widetilde{M}}$.

$\mathbb{M}, M, \widetilde{M}$ **are all bibundles.**

Worldsheet picture

There is a correspondence between Courants and string phase spaces [Alekseev Strobl 05]

$$\text{exact Courant algebroids } [(T \oplus T^\star)M, H] \leftrightarrow \mathrm{Maps}(S^1 \to T^\star M)$$

This was later generalised to a **QP mfold/brane phase space correspondence** [ASA 21], convenient language for us. (In principle known to Pavol Ševera.)

Diagram of symplectic reductions

$$\mathrm{Maps}(S^1 \to T^\star \mathbb{M})$$

$$\mathrm{Maps}(S^1 \to T^\star M) \longleftarrow \mathrm{Maps}(S^1 \to T^\star \mathbb{M}) \longrightarrow \mathrm{Maps}(S^1 \to T^\star \widetilde{M})$$

whence a *lagr. correspondence* between $T^\star \mathcal{L}M = \mathrm{Maps}(S^1 \to T^\star M)$ and $T^\star \mathcal{L}\widetilde{M}$.

At the top we have the phase space of a doubled string ($4d$ coordinates total)!

Summary/remarks

We produced some new examples of topology change, namely ones in Poisson-Lie T-duality with spectators. Duality via (dg-) symplectic reductions of the associated exact Courant algebroids/QP-manifolds, that produce symplectic reductions of the string sigma models.

In our setup, duality sends a principal **bibundle** (obeying 'topological factorisation')

$$G \hookrightarrow M \twoheadrightarrow B \qquad (G \text{ is a (nonabelian) Poisson-Lie group})$$

to another such $\widetilde{M}$ whose fibre $\widetilde{G}$ is the PL dual group.

1. The strange r^{∇} connection-esque 1-form looks suspiciously like a *2-connection* [Aschieri Cantini Jurčo 03]? CS terms for such? Relation to non-abelian gerbes?
2. We have a Hori formula (transformation of RR fluxes). Lift to K(K)-theory? Perhaps as a *(non)commutative correspondence* [Brodzki, Mathai, Rosenberg, Szabo 07]
3. Can do the same dance for M-theory: we know the QP targets [ASA 18, ASA Malek Tennyson 22] corresponding to M- and D-branes. **Topology change in U-duality**?

Thank you!