The Higher Dimensional Tropical Vertex

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Gromov–Witten theory

- **Enumerative algebraic geometry**: counting geometric objects.
  - **Gromov–Witten invariants**: counts curves in a space X, satisfying a given list of constraints.

Example

For any $d > 0$, the number of rational curves in $X = \mathbb{CP}^2$ of degree d, passing through $3d-1$ marked points $p_1, \ldots, p_{3d-1}$ is a Gromov–Witten invariant.

d	1	2	3	4	5	6
$GW(\mathbb{CP}^2)^{0,d}_{p_1,\ldots,3d-1}$	1	1	12	640	84000	$26 \cdot 10^6$

Table: Gromov–Witten invariants of $\mathbb{CP}^2$ for degree $d \in \{1, \ldots, 6\}$.

Gromov–Witten theory

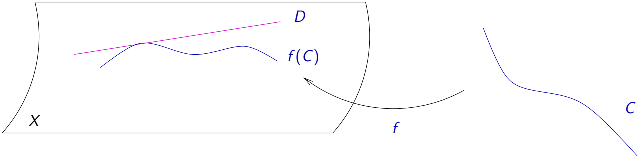
Definition (Kontsevich, 1994)

Let $u = (g, \beta, n)$ with g and n nonnegtaive integers and $\beta \in H_2(X, \mathbb{Z})$. A stable map to X of class u is a morphism $f : (C, x_1, \cdots, x_n) \to X$, such that:

- *C is a projective curve of arithmetic genus g with at worst nodal singularities.*
- *$x_1, \cdots, x_n$ are smooth marked points on C.*
- *The homology class of $f(C)$ is β.*
- *The automorphism group of the map f is finite.*

- Kontsevich/Behrend–Fantechi: The moduli space $\overline{M}_u(X)$ of stable maps to X of class u is a "compact" and "virtually smooth" space.
- Gromov–Witten invariants of X are obtained by integration of cohomology classes over a "virtual fundamental class" of $\overline{M}_u(X)$.

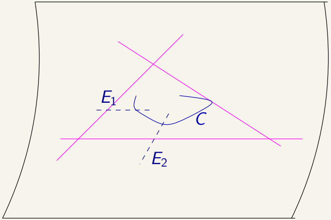
- Log Gromov–Witten theory of Abramovich–Chen–Gross–Siebert is a generalization of Gromov–Witten theory in which X is replaced by a pair (X, D), where X is a smooth projective variety and $D \subset X$ is a reduced normal crossings divisor.
  - If furthermore $K_X + D = 0$ we call (X, D) a **log Calabi–Yau pair.**
- **(Log) Gromov–Witten invariants** of a log Calabi–Yau pair (X, D) are counts of complex algebraic curves contained in X and having prescribed tangency conditions with D.



Example of log Gromov–Witten invariants: $\mathbb{A}^1$ curves in (X, D)

Example

- X: Blow-up of a toric variety X_Σ associated to a fan Σ along hypersurfaces on the toric boundary divisor D_Σ, and $D \subset X$: strict stransform of D_Σ.
- Count of $\mathbb{A}^1$ **curves**, i.e. stable rational maps in (X, D) with image touching D at a single point) define log Gromov–Witten invariants.



The blow up of the projective space $\mathbb{CP}^2$ at two points on D.

- E_1, E_2: exceptional curves
- C: the strict transform of the line passing through the two points we blow up

- Motivation: Mirror symmetry for log Calabi–Yau pairs
  - Gross–Siebert: Construction of the mirror to (X, D) from the data of all $\mathbb{A}^1$ curves.

A wall-crossing algorithm computing $\mathbb{A}^1$-curves for blow-ups of toric varieties in any dimension!

Argüz–Gross[1]

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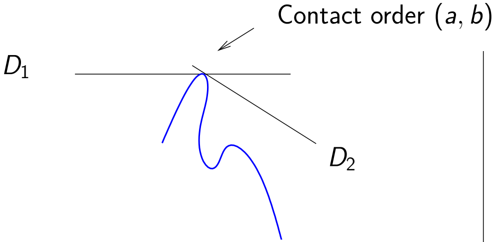
[1] Argüz–Gross, "The higher dimensional tropical vertex", arxiv:2007.08347, **Journal of Geometry & Topology**

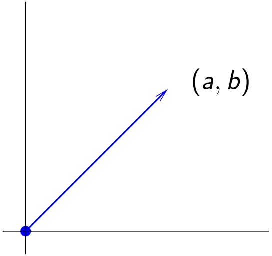
Input: Data of (tropical analogues of) all exceptional curves

↓ Wall crossing algorithm in the tropicalization of (X, D)

Output: Data of all $\mathbb{A}^1$ curves

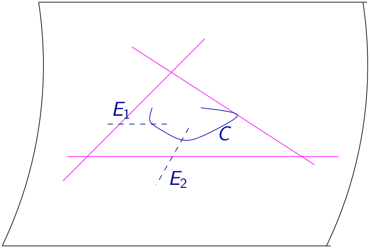
- There is a discrete space $\Sigma_{(X,D)}$, called the *tropicalization of* (X, D), parametrizing possible tangency conditions at a point of intersection of the curve with D.
  - If D is smooth, we just have $\Sigma_{(X,D)}(\mathbb{Z}) = \mathbb{N}$.
  - If D has two components meeting transversally at a nodal point, then $\Sigma_{(X,D)}(\mathbb{Z}) = \mathbb{N}^2$.

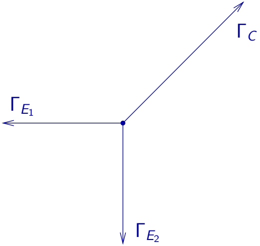




$D = D_1 \amalg D_2$

Tropical $\mathbb{A}^1$ curves





- The *tropical curve* associated to a curve C is the *dual graph* Γ_C:

$$\text{Irreducible components of } C \Longleftrightarrow \text{Vertices of } \Gamma_C$$
$$\text{Nodal points of } C \Longleftrightarrow \text{Edges of } \Gamma_C$$
$$\text{Marked points of } C \Longleftrightarrow \text{Legs of } \Gamma_C$$

- The data of "all" tropical $\mathbb{A}^1$ curves of (X, D) is encoded in a "consistent wall structure" (generalization of a toric fan for non-toric varieties).

Wall structures

- $N \cong \mathbb{Z}^n$, $M = \mathrm{Hom}(N, \mathbb{Z})$.
- $\mathfrak{g}$: Lie algebra and G the associated group.

Definition

A wall structure (scattering diagram) is a collection of tuples $(\mathfrak{d}, f_{\mathfrak{d}})$, where

- $\mathfrak{d}$: codim one subset of $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n$, called a **wall**
- $f_{\mathfrak{d}}$: an element of G, called a **wall crossing transformation**

Example

- $\mathfrak{g}$: Lie algebra of zero-divergence vector fields on a family of tori, $\mathfrak{g} = \bigoplus_{m \in M, n \in N, (m,n)=0} \mathbb{C}[[t]] z^m \partial_n$ with the Lie bracket given by $[z^m \partial_n, z^{m'} \partial_{n'}] = z^{m+m'} \partial_{(m',n)n' - (m,n')n}$
- G: Volume preserving automorphisms of the family of tori

The initial scattering diagram

- We define **an initial wall structure** associated to a log Calabi–Yau pair (X, D) obtained as a blow-up of a toric pair (X_Σ, D_Σ) along hypersurfaces $H_i \subset D_i$ in the toric boundary of X_Σ, we define
- **Walls** : $\mathfrak{d}_i := \mathrm{Def}(\Gamma_{H_i})$ and the attached functions are of the form $1 + 1 + t_i z_i$. Here, $z_i = z^{m_i}$ where m_i is the direction of the ray corresponding to D_i and t_i is a formal variable.

Example: $\mathbb{P}^3$ with two lines

Example

Let $X_\Sigma = \mathbb{P}^3$ and $D_\Sigma = D_1 + D_2 + D_3 + D_4$ its toric boundary. Take H to be the union of lines $\ell_1 \subset D_1$ and $\ell_2 \subset D_2$, and set

- $X := Bl_{\ell_1 \cup \ell_2}(\mathbb{P}^3)$
- D : strict transform of D_Σ.

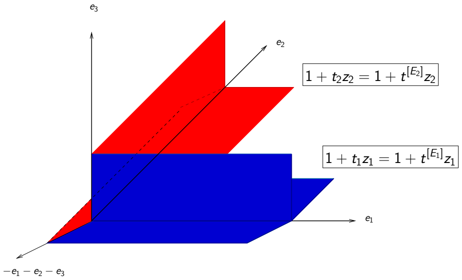
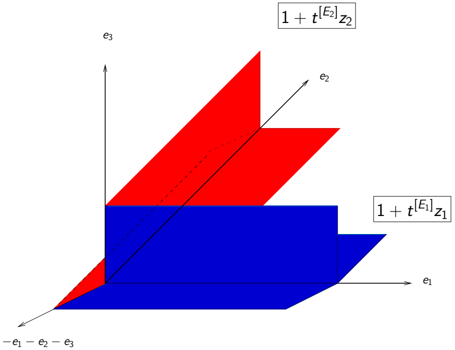


Figure: Walls of the initial wall structure for $\mathbb{P}^3$ with $\ell_1 \subset D_1$ and $\ell_2 \subset D_2$.

- **Crossing walls**: To a wall $(\mathfrak{d}, f_{\mathfrak{d}})$ with a normal vector n, associate a wall crossing automorphism $z^{m_i} \mapsto f^{\langle m_i, n\rangle} z^{m_i}$.



$z_1 = z^{(1,0,0)}$

$z_2 = z^{(0,1,0)}$

$z_3 = z^{(0,0,1)}$

$z_4 = \frac{1}{z_1 z_2 z_3}(1 + t^{[E_1]} z_1)(1 + t^{[E_2]} z_2)$

The wall-crossing algorithm

A wall structure is called **consistent** if for any joint (codim 2 locus where walls intersect), the composition of all wall-crossing transformations on all adjacent walls are identity.

- The initial wall structure is not consistent!

- **Algorithm**: Systematically insert walls $(\mathfrak{d}_{\mathrm{out}}, f_{\mathfrak{d}_{\mathrm{out}}})$ to complete the initial wall structure to a consistent wall structure.
  - This is a purely algebraic algorithm.
  - We wrote a computer algebra code using magma, together with Tom Coates.

The wall-crossing algorithm

$\mathscr{W}$	$f_{\mathscr{W}}$
$\langle e_1, e_2\rangle, \langle e_1, e_3\rangle, \langle e_1, e_4\rangle$	$1 + t_1 z_1$
$\langle e_2, e_1\rangle, \langle e_2, e_3\rangle, \langle e_2, e_4\rangle$	$1 + t_2 z_2$
$\langle e_3, -e_1\rangle, \langle e_4, -e_1\rangle$	$1 + t_1 z_1$
$\langle e_3, -e_2\rangle, \langle e_4, -e_2\rangle$	$1 + t_2 z_2$
$\langle -e_2, -e_1 - e_2\rangle, \langle -e_1, -e_1 - e_2\rangle, \langle e_3, -e_1 - e_2\rangle, \langle e_4, -e_1 - e_2\rangle$	$1 + t_1 t_2 z_1 z_2$
$\langle e_1, -e_2\rangle$	$1 + t_2 z_2 + t_1 t_2 z_1 z_2$
$\langle e_2, -e_1\rangle$	$1 + t_1 z_1 + t_1 t_2 z_1 z_2$

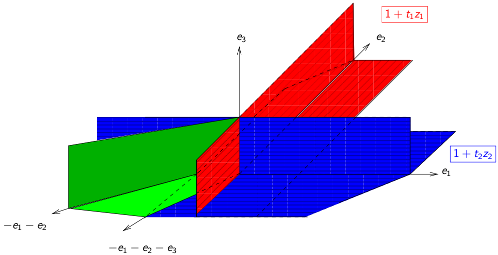


Figure: Walls of the consistent wall structure assoiated to $\mathbb{P}^3$ with two lines

Main result: The higher dimensional tropical vertex

Theorem (Argüz–Gross)

Let (X, D) be a log Calabi-Yau pair obtained from a toric pair (X_Σ, D_Σ) by a blow-up, and let $\dim X = n$. In the associated consistent wall structure, let $(\mathfrak{d}_{out}, f_{\mathfrak{d}_{\mathrm{out}}})$ be a wall produced by the algorithm. Then,

$$\log f_{\mathfrak{d}_{out}} = \sum_{\tau} \sum_{\beta} k_\tau N_{\tau,\beta}(X, D) t^\beta z^{-u_\tau}$$

- *τ is a tropical curve with one leg having direction vector u_τ, and a $(n-2)$-dimensional deformation space which forms $\mathfrak{d}_{out}$,*
- *k_τ is a constant number associated to τ,*
- *$\beta \in H_2(X, \mathbb{Z})$,*
- *$N_{\tau,\beta}(X, D)$: count of $\mathbb{A}^1$ curves of class β and tropicalization τ.*

- Major application: Explicit equations for mirrors to log Calabi–Yau pairs (arXiv:2109.08664).

End

Thank you for your attention!